NOTES ON PERELMAN'S SECOND PAPER

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Contents

Introduction

The manifolds we study here are all orientable, of dimension three.

We use I-X.Y to denote a result in Perelman's first paper, [15]; II-X.Y to denote a result in his second paper, [16]. We follow the notations as listed in the beginning of [16], e.g. $P(x, t, r, \Delta t)$ is a parabolic neighborhood. For example, in page 1 of [16] the strong ϵ-neck is a parabolic neighborhood $P(x,t,\epsilon^{-1}r,-r^2)$ that is ϵ-close to the corresponding part of an evolving $S^2\times\mathbf{R}$, after rescale.

Needless to say, the readers are reading [16], and [15]; so we did not write a complete text with all the definitions, notations, statements; the readers will find all the terminology and statement of theorems in [16], [15].

Some of *my* incorrect and inaccurate interpretations, or arguments, in this notes were spotted and corrected. But it is very likely that many more are still there. So corrections and comments are very welcome!

From a logical point of view, the results of the second paper, [16], depends on the following sections of the first paper, [15]: I-7, I-8, I-9, I-10, [1] I-11, I-12.1. A nice set of notes for the first paper was written by Kleiner and Lott; see [14]. Section II-8 depends on I-1 and I-2; however we will not get into II-8, because the major topological results and arguments are already given in II-7.

Here is a brief overview of what is going on:

Combine with the results from (especially, Sec. I-12 of) the first paper [15], in section II-1 Perelman gave a rough classification of noncollapsing ancient solutions with positive, bounded curvature. Using this, one gets a good picture of the regions at which the scalar curvature R is sufficiently big: near such point the manifold looks like either a tube, or a cap, or a compact manifold with positive curvature that is classified by Hamilton's work [8]. These possibilities, especially the tubes and the caps, are called the **canonical neighborhoods**.

In particular, in section II-3 Perelman pointed out that the singularity looks like the tip of a horn. The other end of this horn may be another horn, or a cap, or a noncollapsed manifold. The idea of surgery is, wait till singularity happens, then break the tip of the horn and glue in a cap, i.e., smoothing off the horn tip. *Genericly*, the reason we get a horn as singularity is, the metric pinched off at the center of a long, thin tube at the singularity time, and thus produces a pair of horns. So surgeries are just breaking off connect sums, or remove an $S^2\times S^1$ from the prime decomposition. In particular, we can recover the topological information.

After each surgery, the manifold may become disconnected. We throw away any connected components that are diffeomorphic to a space form with positive curvature, and consider the remaining components individually.

In section II-2 Perelman constructed a standard solution; that is, the **added cap** which we want to glue in during a surgery. Roughly speaking, we need some control of the added cap. In particular, after the surgery time, we want the added cap looks like a cap (i.e. the standard solution) for at least a short time period; that will be proved in II-4.5 to II-4.7.

The major goal of Section II-4 and II-5 is to prove the canonical neighborhood assumption, namely, even with surgeries, one must get canonical neighborhoods at high scalar curvature regions. Moreover, surgeries does not effect the curvature pinching of Hamilton-Ivey, [12], [13], which says that at high scalar curvature region the negative part of the curvature is relatively small compare with the positive part, and this improves when time goes to ∞. These two conditions are the cornerstone of all the arguments.

The proof of these results will be completed at the end of Section II-5. Many compactness arguments are involved. Roughly, the canonical neighborhood assumption has some strong implications, e.g., the solution is noncollapsing, the added cap will look like a cap for a while after the surgery, as mentioned above, etc. One now argues by contradiction, assume there is an earliest time that the canonical neighborhood assumption fails, then we try to get a limit for such counter examples: the

[1] I don't know where the main theorem of I-10 was used; although Claim 1 and Claim 2 (those are elementary) are widely used.

limit turns out to be an ancient solution which automatically have canonical neighborhoods, that is a contradiction.

However, as time goes by, the parameters for the canonical neighborhood may get weaker; e.g. how noncollapsed the solutions are, and when the scalar curvature R are regarded as being *big*.

We may keep on performing surgeries indefinitely. But the surgery times are discrete, because within finite time intervals we remove at least a definite amount of volume at each surgery.

Section II-6 gives some crucial estimate when time t_0 is big, and scale r_0 is *relatively* small compare with $\sqrt{t_0}$. For example, Corollary II-6.8 says, roughly, after running the Ricci flow with surgery for quite a long time, if at t_0 a ball of radius r_0 has curvature bounded below by $-r_0^{-2}$, and the volume has a lower bound, then there is no surgery on this ball shortly before t_0; and we can estimate upper bound of curvature.

The motivation is in I-11.4-6. Think, if we don't have an upper bound in curvature, then do a rescale limit, we expect an ancient solution. The lower bound in volume tells us the ancient solution should have Euclidean volume growth. But as we know (e.g. I-11.4 and so on) ancient solutions look like a tube at infinity. Although the radius of these tubes can grow unbounded, the solution cannot have Euclidean volume growth. That is a contradiction.

These will be used in Section II-7. Let $\rho(x,t)$ be the biggest $\rho > 0$ so that on $B(x,t,\rho)$,

$$Rm \geq -\rho^{-2}.$$

One want to check if the ball $B(x,t,\rho)$ collapses, i.e.

$$\mathrm{Vol}(B(x,t,\rho)) < w\rho^3,$$

for some small constant w. It turns out, if ρ is small compare with $\sqrt{t}$, the ball must collapse; one can use Corollary II-6.5: noncollapsing implies a curvature bound. But now, $-\rho^{-2}$, the lower bound in sectional curvature, is too negative and the pinching estimate ask for an even bigger scalar curvature.

If ρ is not small compare with $\sqrt{t}$, it might collapse; if it does not collapse, using a differential inequality argument similar to the one in Hamilton's nonsingular paper, [12], Perelman showed that after rescale, the metric on that piece converges to hyperbolic metric.

Roughly, this gives a thin-thick decomposition when time is big; The thick parts converges to hyperbolic pieces and the thin parts are collapsed. It is used here some of Hamilton's arguments in his nonsingular paper [12], to show the hyperbolic pieces persist and their fundamental groups inject into the fundamental group of the whole manifold.

Finally in II-7.4 Perelman claimed that the thin part is a graph manifold, thus implied the Thurston geometrization conjecture! This graph manifold result seems to be known to Perelman many years ago; but it was not published. However, in a recent preprint [22] of Shioya and Yamaguchi, they gave a proof for this result.

We remark that in Perelman's third paper, [17] (and the paper of Colding and Minicozzi [6]), among other things, it was proved that after sufficiently long time, all components of the solutions are prime. So miraculously, the Ricci flow realizes the classical prime decomposition.

1. Section II-1.2

Here Perelman rules out κ-noncollapsed, noncompact, 3-dimensional shrinking solitons with strict positive, bounded curvature. The proof is well written, and is also explained in [14].

Remark 1.1. *The first two paragraphs in the proof also rules out κ-noncollapsed, noncompact,* 2*-dimensional shrinking solitons with strict positive curvature. In fact there is a positive lower bound in sectional curvature and the manifold must be compact.*

If we don't know the curvature is bounded, one can argue using the disc picking argument in I-10, and get a contradiction because splitting 2*-manifolds are flat. Compare with I-11.4.*

In particular, there are only two types of κ-noncollapsed, 3-dimensional shrinking solitons with nonnegative curvature:

1). S^3/Γ with standard metric, i.e. a space form with positive curvature.

2). $S^2 \times \mathbf{R}$, or $S^2 \times \mathbf{R}/\mathbf{Z}^2$, where the group $\mathbf{Z^2}$ action is

$$(x,t) \mapsto (-x,-t).$$

Topologically, $S^2 \times \mathbf{R}/\mathbf{Z}^2$ is $\mathbf{R}_+ \times S^2$ with the end glue to $\mathbf{RP}^2$ by projection $S^2 \to \mathbf{RP}^2$; or, in another word, remove a closed ball B^3 from the project space $\mathbf{RP}^3$.

In fact, 1) is the only case with positive curvature. The compact case is proved by Ivey [13]; the noncompact case is ruled out here in II-1.2. When the curvature is not strictly positive, by Hamilton's maximum principle, the solution splits. Locally it is $\mathbf{R}$ times a two dimensional ancient solution. By I-11.3, the only two dimensional noncollapsing ancient solution is S^2 with the standard metric. So our solution is $S^2 \times \mathbf{R}$ or its quotients. If the quotient is compact, let time go to $-\infty$, we see only the S^2 direction expands, so the solution is not κ-noncollapsing (it is not a soliton anyway). On the other hand, all possible noncompact quotients of $S^2 \times \mathbf{R}$ are listed in 2).

2. Section II-1.3

Roughly we now have a classification of κ-noncollapsed ancient solutions with nonnegative, bounded curvature in dimension 3. We write down the classification here by looking at their asymptotic soliton (compare I-11.2).

Case one: The asymptotic soliton is S^3/Γ.

Then by [8], the ancient solution must be S^3/Γ itself, with constant positive curvature.

Case two: The asymptotic soliton is $S^2 \times \mathbf{R}$, or $S^2 \times \mathbf{R}/\mathbf{Z}^2$. We have :

2a). The solution does not have strictly positive curvature. Then as before (I-11.3), we see the solution is $S^2 \times \mathbf{R}$, or $S^2 \times \mathbf{R}/\mathbf{Z}^2$ by the strong maximum principle.

2b). If the solution has strictly positive curvature and not compact, we know it is diffeomorphic to $\mathbf{R}^3$ by the soul theorem; it can not have Euclidean volume growth by I-11.4. In fact, one can show that at infinity the solution looks like a tube; this is based on I-11.7 and an application of the splitting theorem.[2]

2c). If the solution has strictly positive curvature and compact. We assume the curvature is not constant; i.e. not as in case one. We know it is diffeomorphic to a space form by [8]. We claim in this case it could only be S^3 or $\mathbf{RP}^3$:

Lemma 2.1. *If a κ-noncollapsing ancient solution has positive bounded curvature and is compact, and the curvature is not constant, then it must be diffeomorphic to S^3 or RP^3.*

Proof. Since the solution, M, is not of constant curvature, we know the asymptotic soliton can only be $S^2 \times \mathbf{R}$, or $S^2 \times \mathbf{R}/\mathbf{Z}^2$.

This means, at some ancient time t_α, there is a long tube in the solution. Since $\pi_1(M)$ is finite, if we break the tube at one center sphere S^2 the solution will be disconnected: it became two manifolds M_1^α and M_2^α.

[2]Here in dimension 3, it is conjectured the radius of the tube goes to ∞ when we travel away from the base point. In another word, $R \to 0$ at the space infinity.

We observe, that the structure of M_1^α is, a long tube connect to a compact manifold E_1^α; if we rescale the scalar curvature at **any** point of E_1^α to 1, the diameter of E_1^α is bounded by a universal constant D. This is basically proved in I-11.8; except that, there is a possibility that another tube connects to E_1^α. But this is ruled out by the following principle:

A compact set E in the sense of I-11.8 can not connect to two tubes.

Otherwise, E is now on the middle of a long minimal geodesic. Using a compactness argument based on I-11.7, combine with the Toponogov-Cheeger-Gromoll splitting theorem, we see E itself must be a tube.

Take M_1^α. Assume p^α is the point on M_1^α that has maximal distance to the S^2 at which we broke the tube. So $p^\alpha \in E_1^\alpha$. We rescale the curvature at p^α to be 1 and take a pointed Gromov-Hausdorff limit at p^α when $t_\alpha \to -\infty$, this is possible by I-11.7. So by I-11.8 we see no topology of M_1^α will "slide to ∞" when we take limit. In particular, M_1^α will be diffeomorphic to the limit for α sufficiently big.

By construction, we know the structure of M^∞ is to attach a half tube to a compact manifold E_1^∞ with a definite diameter control. In fact, the proof of the soul theorem tells us, the topology must all be on one side of the tube; one checks that the totally convex set in that proof can not contain any tube points. So E_1^∞ contains all the topological information of M_1^∞.

The limit, (M_1^∞, p^∞) is clearly a complete manifold with nonnegative curvature. And p^∞ can not be on a tube by construction. It is itself an ancient solution. Now apply the soul theorem. Notice $\pi_1(M_1^\infty)$ is finite, so the soul can only be a point, or S^2, or $\mathbf{RP}^2$. That is, it is diffeomorphic to one of the following:

1). $\mathbf{R}^3$;

2). $S^2 \times \mathbf{R}$;

3). $\mathbf{RP}^3 - B^3$.

2) is ruled out because p^∞ can not lie on the middle of a tube by the choice of p^α. So we only have 1) and 3) as possible M_1^∞.

However, the limit of M_1^α and M_2^α can not be 4) at the same time, because that implies $\pi_1(M) = Z_2 * Z_2$ is infinite. So at least one of M_1^α and M_2^α is diffeomorphic to 1); and the other diffeomorphic to either 1) or 4). That proves our lemma. $\square$

So the structure of this solution may be two caps connected together, and at least one of the caps is B^3.

3. II-1.4

All the cases, 1, 2a, 2b in the previous sections have examples. The example for 2b is the Bryant soliton (Bryant himself did not publish a paper about this soliton, but one can read the papers of Ivey for some examples).

In this section an example was given for case 2c. Compare with the arguments in II-2, Claim 4.

4. II-1.5. Definition of canonical neighborhoods

In dimension 3,

Lemma 4.1. *There exist κ_0, so that for all κ, any κ-noncollapsing ancient solution with bounded curvature is either a κ_0-solution, or a metric quotient of the round sphere.*

Proof. If the ancient soliton is a space form, we know the solution is itself a space form (that could be very collapsed).

Otherwise, the ancient soliton contains a tube. We follow an argument in I-7.3; if the solution is very collapsed at p, we get a small $\tilde{V}$. As in the proof of I-11.2, there is a curve from p to a very ancient point q where $l \leq 3/2$; and near q the solution looks like a tube; the curvature of which also behave like a tube. So near q we get a definite lower bound in $\tilde{V}$. That contradicts to the monotonicity of $\tilde{V}$. □

Remark 4.2. *Here one should notice, the quantity $\tilde{V}$ is invariant under simultaneous rescale of distance and time, i.e. $\tilde{V}$ is a quantity with no dimension. In particular we can compute $\tilde{V}$ at an ancient time using bounded τ, say $\tau \sim 1$, thanks to the rescale allowed by I-11.2. This is however not possible in general situations, there because of the factor $\tau^{-n/2}$ in $\tilde{V}$, one gets very weak conclusion of noncollapsing when time is very big.*

Then we have inequality (1.3):

Lemma. *On κ-solutions $|\nabla R| < \eta R^{3/2}$, $|R_t| < \eta R^2$.*

Proof. This is automatic for space forms.

On the other hand, by the proof of I-11.7 we know curvature stay finite within finite distance for a κ-solution; the bounds depend on κ. Now we know apart from space forms, κ-solutions are κ_0-solutions; so we can bound curvature on a ball with canter (x_0, t_0) of radius $R(x_0, t_0)^{-1/2}$ by

$$R < CR(x_0, t_0).$$

The Harnack inequality tells us this is also a bound on a parabolic neighborhood. So we can use Shi's gradient estimate, [18], [19]. (Note R_t is bounded by $|\nabla\nabla R|$ and R since $Rm \geq 0$). □

Remark 4.3. . *The η in (1.3) does not depend on κ.*

Remark 4.4. *In* [16], *it was not written explicitly that an ϵ-cap should be ϵ-close to a piece of ancient solution; it seems one can impose this requirement as we will do below, or one just follow* [16]. *What is important are the following properties of any piece E of ancient solutions:*

i). If E contains a minimal geodesic of length ϵ^{-1}, and $R = 1$ at the midpoint p of this geodesic, then it contains a tube centered at p.

ii). If E does not contain any tube, then $\operatorname{Diam} E < \epsilon^{-1}/2$ *if we rescale the scalar curvature R at* **any** $p \in E$ *to* 1.

iii). The gradient estimate II-(1.3) works on E.

For ancient solutions, i) is proved by compactness: if not, we take a limit of counterexamples, this is possible by I-11.7. The limit splits, by the Toponogov-Cheeger-Gromoll splitting theorem. So the limit itself is a tube (by I-11.3). That is a contradiction. Here certainly ϵ^{-1} should be sufficiently big. This type of **almost splitting theorem** *is very useful.*

ii) by I-11.8;

iii). is just proved above.

It is these properties that are used in the later proofs.

Finally, Perelman discussed (although not explicitly mentioning the name) the **canonical neighborhood** in II-1.5. This is a very important notion, so we state it here:

There exist a number $r > 0$, so that for any κ-noncollapsing solution [3] of Ricci flow without surgery on compact manifold M with $|Rm| \le 1$ at time 0, whenever

$$R(x,t) > r^{-2},$$

at some point (x,t), we have one of the following possibilities:

(a). We rescale the parabolic neighborhood, $P(x,t,\epsilon^{-1}R(x,t)^{-1/2}, -R(x,t))$ by $R(x,t)$ and do the corresponding time-rescaling, and move t to 0; so we get a parabolic neighborhood $P(x,0,\epsilon^{-1},-1)$.

Then $P(x,0,\epsilon^{-1},-1)$ is ϵ-close to $P(p,0,\epsilon^{-1},-1)$, here $P(p,0,\epsilon^{-1},-1)$ is a parabolic neighborhood in the standard evolving $S^2 \times \mathbf{R}$ which has scalar curvature 1 at time 0. Here and below, ϵ-close is in $C^{\epsilon^{-1}}$ topology.

This possibility is called a **strong ϵ-neck**.

(b). We rescale the ball $B(x,t,\epsilon^{-1}R(x,t)^{-1/2})$ by $R(x,t)$ so we get a ball $B(x,t,\epsilon^{-1})$.

Then the ball $B(x,t,\epsilon^{-1})$ is ϵ-close to a piece of an ancient solution (at some time) and is diffeomorphic to either the solid ball B^3 or $\mathbf{RP}^3 - B^3$. Moreover, the set $B(x,t,\epsilon^{-1}) - B(x,t,C)$ is ϵ-close to a piece of $S^2 \times \mathbf{R}$ with constant scalar curvature $C^{-1} < R < C$.[4]

This possibility is called a **ϵ-cap**.

(c). The ball $B(x,t,\epsilon^{-1}R(x,t)^{-1/2})$, after rescale the metric by $R(x,t)$, is ϵ-close to an ancient solution (at some time) and is diffeomorphic to S^3 or $\mathbf{RP}^3$.

(d). The ball $B(x,t,10R(x,t)^{-1/2})$, after rescale the metric by $R(x,t)$, is ϵ-close to a manifold with constant sectional curvature $1/6$.

Neighborhoods defined by (a), (b), (c), (d) are called **canonical neighborhoods**. However, to prepare for the surgeries, we also want to include in the definition of canonical neighborhoods one more possibility (e), which is similar to (b):

(e). We rescale the ball $B(x,t,\epsilon^{-1}R(x,t)^{-1/2})$ by $R(x,t)$ so we get a ball $B(x,t,\epsilon^{-1})$.

Then the ball $B(x,t,\epsilon^{-1})$ is diffeomorphic to the solid ball B^3. Moreover, the set $B(x,t,\epsilon^{-1}) - B(x,t,\epsilon^{-1}/2)$ is ϵ-close to a piece of $S^2 \times \mathbf{R}$ with constant scalar curvature $C_1^{-1} < R < C_1$; the whole ball $B(x,t,\epsilon^{-1})$ is ϵ-close to the standard solution (Defined in II-2) at some time $t < 1$; after suitable rescale.

Remark 4.5. *The reason I add this possibility (e) is, the standard solution is generally* **not** *close to an ancient solution; see the discussion of II-2-Claim 5 in this notes.*

It seems possibly to relax the definition of canonical neighborhoods a little, for the caps; we can allow metrics other than we mentioned above as long as the metric on the cap is of nonnegative curvature, close to a cylinder near boundary, satisfying gradient estimate of R and the almost splitting theorem, i.e. "point on a long geodesic lies on a tube". I learned this from Prof. Lott.

In fact there exists a **canonical neighborhood parameter** $r > 0$ so that the above is true for *smooth* solutions; (e) is not necessary in this case. This is proved by I-12.1 and the classification of ancient solutions; as we have seen from II-1.3.

For example, we explain that category (b) appears in the following situation: By I-12.1 we know near (x,t) there is a neighborhood that is close to a piece of an ancient solution; we assume it does not fall into categories (d), (a), (c). So the ancient soliton is $S^2 \times \mathbf{R}$. Now since $B(x,t,\epsilon^{-1})$ is quite big (diameter about ϵ^{-1}), it contains a long minimal geodesic; near the middle of the geodesic the

[3]Not just ancient solutions; thanks I-12.1. κ-noncollapsing is alway true for some κ in finite amount of time by I-7.3.

[4]C is the constant in I-11.8.

ancient solution looks like a tube by a compactness argument; as we have seen in II-1.3. Since we ruled out case (a), (x,t) is not in this tube. So (x,t) lies in a compact set E in the sense of I-11.8, that compact set has diameter smaller than $\epsilon^{-1}/2$ by I-11.8. The remaining part, $B-E$, must *all* be a tube because: i. we ruled out case (c); and ii. on a single ancient solution one can not have a tube connect to a compact set in the sense of I-11.8 which leads to *another* tube, see the discussion of II-1.3 of this notes.

Remark 4.6. *I don't know precisely the possible shapes of a cap. But roughly, its head can not be "fat".*

Precisely, let's rescale the scalar curvature at any point on the cap to be 1. *Remember the part of a cap that is not on a tube is a compact set* E *in the sense of I-11.8; it is attached a tube with length roughly* $\epsilon^{-1} \gg C$, *where* C *is the constant in I-11.8. So let* r_0 *be the radius of the tube and* r_0 *is comparable to* 1, *as we have seen above. "Not fat" means the following:*

The volume of any ball of radius up to Cr_0 *is no more than* $1+\epsilon'$ *times the volume of a ball with same radius on the tube.*

The proof uses volume comparison with base point far away on the tube. Yau used this argument to show that a noncompact manifold with nonnegative Ricci curvature has at least linear volume growth. See [23] *for an account of this.*

About the final remark in II-1.5: we should point out, in case (c), the whole manifold is a compact set in the sense of I-11.8. One might think that it could be that caps being connected by a long tube; as we have seen in II-1.3, but in that case we don't classify it to (c), we classify it to made up of two pieces of type (b) and one or more pieces of (a). So the scalar curvature bound follows from I-11.8.

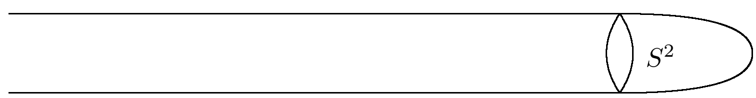
The volume bounds for (a), (b) follows from the fact that, the neighborhood contains a piece of tube.

The volume bound for (c) follows from κ_0-noncollapsing of ancient solutions that is not space forms.

The sectional curvature lower bound for (c), follows from a compactness argument: if not, we can take limit for counter examples; then the limit splits. Thus we see some counterexample contains a tube; so we should not have classified the solution into category (c).

5. Section II-2

In this section Perelman constructed a standard solution. This is a solution of the Ricci flow, whose initial data looks like a cap, and shapes like a cylinder at infinity:



We will give a more precise construction in our review of II-4.4, for now it is enough to know that at time 0 the solution is like the above picture, it is radial symmetric and with nonnegative curvature.

By Shi's theorem ([18], Theorem 1.1), we know the standard solution exits for a while, and we have curvature control.

Exercise 1. *Prove, under suitable assumptions, if the curvature operator is nonnegative at $t = 0$ on a noncompact manifold, then the (unique ?) solution of the Ricci flow preserves this.*

Answer: See [19], Theorem 4.14. It is not trivial.

Claim 1.

I prefer to work with forms, i.e. the dual of vectors.

For any one form $u(t)$ that evolves under the Ricci flow by

$$u'(t) = \Delta_t u + b,$$

where $\Delta = \nabla^2_{e_i,e_i}$, b is a (time dependent) one tensor, we compute

$$\begin{aligned}(\nabla u)'_t(X,Y) =&\Delta\nabla u(X,Y) + \nabla_X b(Y) + \nabla_X \operatorname{Ric}(Y,e_i)\cdot u(e_i)\\ &+ 2\nabla_{e_i}u(e_j)\cdot R(e_i,X,Y,e_j) - \nabla_{e_i}u(Y)\cdot \operatorname{Ric}(X,e_i).\end{aligned}$$

Here $\nabla u(X,Y) = \nabla_X u(Y)$; and $R(e_i,e_j,e_j,e_i)$ is the sectional curvature (opposite to Hamilton's notation). So if we make[5]

$$u'(t) = \Delta u - \operatorname{Ric}(e_i,\cdot)u(e_i),$$

we get, (let $v_{i,j} = \nabla_i u_j$) for an orthonormal frame,

$$\dot v_{ij} = (\Delta v)_{ij} + 2R_{iklj}v_{kl} - R_{ik}v_{kj} - R_{kj}v_{ik}.$$

Note, unlike the formula in [9], here the frame i,j,k,l is fixed.

We now show that the center of the cap is the unique zero of any Killing field for any time; first, there exists a zero any time: We consider the Busemann function associated to a ray γ:

$$B(x) = \lim_{s\to\infty} d(x,\gamma(s)) - s.$$

Lemma 5.1. *All Busemann functions are the same, up to adding a constant.*

Proof. We need to use II-2, Claim 2, whose proof does not depend on the discussion here. That says at space infinity, the solution looks like a tube.

First assume there are two rays from p, and we want to compare the two Busemann functions associated with these two rays. One can start by showing that all points on an S^2 on the tube far away have almost the same distance from p. When trying to compare the distances from p to $x, y \in S^2$, one gets a very thin triangle pxy with two angles at x and y almost equal to $\pi/2$. Then the Toponogov triangle inequality tells us $d(p,x) \approx d(p,y)$. In particular these S^2 gets more and more close to being level surfaces of the distance function to p. The conclusion then follows easily.

The general case is similar, also because eventually rays go into a tube with definite radius. □

By the proof of the Cheeger-Gromoll soul theorem, we know there is a unique maximal point of the Busemann function. An isometry moves a ray to a ray, in particular, it does not change the Busemann function. So it does not change the unique maximum point. So we have at least one common zero for all the Killing fields.

Let $Z(t)$ be the common zeros of all the Killing fields at time t. So $Z(t)$ is a closed set and $\cup_t Z(t)$ is closed in space-time. Assume p is a limit point of the common zeros of all the Killing fields, when $t \to t_0^+$.

Lemma 5.2. *If u is a Killing form, $u = 0$ at p, then $\Delta u = 0$ at p.*

[5] A typo here?

Proof. Compute using normal frame e_i at p:

$$\begin{aligned}\Delta u(e_j) =&\nabla_{e_i}\nabla_{e_i}u(e_j) = e_i(\nabla_{e_i}u(e_j))\\ =&-e_i(\nabla_{e_j}u(e_i)) = -\nabla_{e_i}\nabla_{e_j}u(e_i)\\ =&-\nabla_{e_j}\nabla_{e_i}u(e_i) + u(R(e_i,e_j)e_i) = -e_j(\nabla_{e_i}u(e_i)) = 0.\end{aligned}$$

□

Very roughly, this tells us zeros of Killing fields moves with speed 0 in space direction.

Now notice, if one direction is not contained in $Z(t)$, there is a Killing field which rotates that direction with a nonzero angle. Combine this with the above lemma, we see

$$\frac{d}{dt}\Big|_{t=t_0}\sup_{z\in Z(t)}(d^t(z,Z(t_0)) = 0.$$

Here a superscript t means we use the distance at time t. We can also run the above in reverse time, so we conclude that $Z(t)$ does not depend on time t. At time 0 it is a single point p, the center (or head) of the cap. So p remains the unique common zero for all Killing fields.

Consider the set $S_r = \{x|d(x,p) = r\}$. It is fixed by all Killing fields. Also the set of isometries act transitively; this is trivial for sufficiently small r, so then it is true for all bigger r.

Moreover, p is a pole: every geodesic from p is a ray, there is no conjugate points.

That proved Claim 1.[6]

We can apply the similar method in studying rays from the pole p, each ray is the zero of a particular Killing field, and *vice versa* (note a Killing field induces a degree 1 map from S_r to itself, so it must have fix points on each S_r). So actually these rays remain to be rays all the time.

Claim 2.

This follows from a compactness argument. We move to infinity in space; and find a limit by Shi's theorem [18], and Hamilton's compactness theorem [10].

The limit has initial data exactly $S^2\times\mathbf{R}$; and we claim that initial data has unique solution; the S^2 factor shrinks to a point at time 1.

This need a curvature bound. It is provided by [18]. Also need the fact that positive curvature operator is preserved, see [19], theorem 4.14.

First, the curvature can not be strict positive by the splitting theorem. By the strong maximum principle[7] the solution splits all the time. Moreover, the split direction is invariant in time. So the solution is just the standard $S^2\times\mathbf{R}$ as we expected.

In particular, the compactness argument tells us the standard solution get singular no later than $t = 1$. In fact, assume some parts survives at time 1, we see near time 1 the infinity is very thin tubes, that violates the volume comparison, as we have seen many times.

Remark 5.3. *A warning: for an n-sphere with constant sectional curvature* 1*, the time it takes to go singular is* $1/(2n-2)$*, not* 1*. See* [11].

A useful byproduct is, we get *explicit* control of all derivatives of curvature, even near $t = 0$. In fact, we have control of these *explicitly* at space infinity by Claim 2, so in particular, we get *some*

[6]It seems one can prove Claims 2, 3 without using Claim 1. Then the uniqueness result, Claim 3, implies Claim 1.

[7]See, by [9] section 8, this strong maximum principle is a local one.

uniform bound on the whole manifold. Then notice the evolution equation (see [8])

$$(|\nabla^d Rm|^2)'_t = \Delta|\nabla^d Rm|^2 - 2|\nabla^{d+1}Rm|^2 + \sum_{i=0}^{d} C_i(\nabla^i Rm, \nabla^{d-i}Rm, \nabla^d Rm),$$

where C_i means a suitable contraction. Now we can carry the maximum principles on compact manifolds (see, for instance, [5], compare with [19]) and get explicit bounds for derivatives of the curvature Rm. This is stronger than the theorem in [18] could offer, it will be used in the proof of the next Claim.

Claim 3.[8]

We use the "linearized Ricci flow"; that is DeTurck's trick, there one modifies the Ricci flow by a certain diffeomorphism so that the Ricci flow becomes a strict parabolic equation. See [2], or the Appendix to this notes for more details. We can do the diffeomorphism thanks to Claim 2.

Precisely, we solve equation (31.8) in the Appendix to this notes.

The important thing is, the linearization of the modified Ricci flow equation (see the Appendix) is now a linear parabolic equation

$$\frac{d}{dt}h_{ij} = \Delta h_{ij} + 2R_{kijl}h_{kl} - R_{ik}h_{kj} - R_{jk}h_{ki}.$$

Here R_{ijji} is the sectional curvature.

Exercise 2. *Why is this equation the same as the Killing field equation in Claim 1?*

Assume $g(t)$ is a solution; and $g(t) + v(t)$ is also a solution, with the same initial data; $v(0) = 0$. Note by the gradient estimate of [18], [19], and our remark at the end of Claim 2, we can control all the derivatives of curvatures of these solutions.

We compute, by integrating the linearization equation,

$$v'(t) = \Delta v_{ij} + 2R_{kijl}v_{kl} - R_{ik}v_{kj} - R_{jk}v_{ki} + \epsilon v_{ij} + \eta_k \nabla_k v_{ij} + \theta_{ab}\nabla^2_{ab}v_{ij},$$

here ϵ, η, θ goes to 0 as $t \to 0$. Pick any set B so that ∂B is almost a standard S^2 far away on the tube. We compute

$$\begin{aligned}
\Big(\int_B |v|^2\Big)' =& 2\int_B v(\Delta v + ... + \epsilon v + \eta_k \nabla_k v + \theta_{ab}\nabla^2_{ab}v + Cv + E_i\nabla_i v) + C(R)|v|^2 \\
\leq& \epsilon + \int_B -2|\nabla v|^2 + Cv^2 + E_i v\nabla_i v + v(\eta_k\nabla_k v + \theta_{ab}\nabla^2_{ab}v) \\
\leq& \epsilon + \int_B Cv^2
\end{aligned}$$

Here $Cv + E_i\nabla_i v$ in the first line comes from variation of volume form; the last term in the first line comes from variation of metric and volume. ϵ on the beginning of the second line comes from the boundary term of integrate by parts and Claim 2; in the last line, all the remaining terms are absorbed into the v^2 integral, or cancelled with the $-2|\nabla v|^2$ integral; one checks this by the standard inequality $2ab < \epsilon a^2 + \epsilon^{-1}b^2$ and integrate by parts; one can use partition of unity here because the metrics are basically the same. So

$$\int_B |v|^2\Big|_t \leq \int_B |v|^2\Big|_0 e^{Ct} + \epsilon\int_0^t e^{C(t-r)}dr = \epsilon\int_0^t e^{C(t-r)}dr.$$

[8]It appears to me that this proof can be done without using Claim 1. The idea behind is the following: By Claim 2, all solutions differ by some 2-tensor h in the Schwartz class $\mathcal{S}$. But if one solves the linearizion equation of the Ricci flow (after doing the DeTurck trick) within $\mathcal{S}$, the solution is unique by standard energy estimate for parabolic equation. Then perhaps several tools are at hand, e.g. implicit mapping theorem, contract mapping theorem, etc, to conclude uniqueness.

Nevertheless, the conclusion of Claim 1 is used in II-4.4 to construct surgery, at least implicitly.

By Claim 2, $\epsilon \to 0$ if we make the set B exhaust the manifold. So

$$\int_B |v|^2\Big|_t = 0,$$

the solution is unique.

Claim 4.

One need to check, the standard solution actually exists on $[0, 1)$.

First we construct the solution via compact manifold: take a long tube and add two caps on the two ends and run Ricci flow. We can make the tube longer and longer and take limit at the center p of one cap, thus we get a solution.

Assume the solution get singular at $T < 1$. We observe, when $d_0(x, p)$ is sufficiently big, in fact the solution extends to any $T' < T$ near x. This is because, by Claim 2, for x far away, the solution evolves like a tube.

But notice, by our construction, the solution is the limit of compact solutions; in particular, the canonical neighborhood assumption (see II-4.1 for definition, and II-1.5, I-12.1 for proof) works. So if we choose T' in the previous paragraph sufficiently close to T, we get a curvature bound near x for all time $t \in [T', T)$ by the gradient estimate II-(1.3). So the solution extends smoothly to T near x.

For fixed T, if singularity appears somewhere, it lies within finite distance to x (the "end" direction looks more and more closer to a tube so there will be no singularity). We now argue using Claim 2 of I-12.1: note the curvature is positive, so near singularity the metric looks like a piece of non-flat cone; this works because the solution is the limit of compact solutions so we can use the canonical neighborhood assumption, e.g. high curvature region at the middle of a minimal geodesic looks like a tube, etc. So we get a contradiction to the strong maximum principle, see [9], or [14] on I-11.4; the argument is local.

Remark 5.4. . *One can also argue using I-10.1, as written in the paper. I don't feel this is unavoidable; in any case we must use the canonical neighborhood assumption to make the argument in I-12.1 work.*

Claim 5.

This claim says that the standard solution satisfies the canonical neighborhood assumption (see II-1.5 and II-4.1). There is a slight abuse of the terminology because here what it means is, the standard solution is close to parts of ancient solutions (thus canonical neighborhood) even if the curvature is not big, e.g. for time near 0. What we have in mind is, in doing surgeries we will glue (the head of) a standard solution to a broken horn; there we have to rescale the standard solution to the surgery scale, h. Then the curvature will be around h^{-2}, which is big, and we want the canonical neighborhood assumption.

One by-product of Claim 4 is, the standard solution will go singular at time 1 *everywhere*, i.e. it is not possible that some parts go singular while other parts remain smooth.

First, we prove the claim for time close to 1, the canonical neighborhood assumption holds everywhere on the standard solution. In fact, when time close to 1 the curvature R must be big everywhere on the standard solution. This is because, first, the standard solution goes singular everywhere near time 1 as we observed, and, second, the standard solution is the limit of compact solutions as in Claim 4, so it satisfies the canonical neighborhood assumption (see II-1.5) and especially the gradient estimate II-(1.3), so we can not have low curvature region near time 1. So the canonical neighborhood assumption for compact manifolds applies everywhere to the standard solution when time close to 1.

For smaller time, we observe the following: There exist $D > 0$ so that if a point q of the standard solution lies at the middle of a minimal geodesic of length at least D, then there is a tube neighborhood of q. Because we now stay away from time 1, this follows from Claim 2.

On the other hand, for the set of points not on a tube, first we get an upper bound for its diameter, by a compactness argument. [9] Then because we work away from time 1, we get *some* curvature bounds, in particular, a bound of $R_{\max}/R_{\min}$. Then we have the gradient estimate II-(1.3) (perhaps we need to increase η in II-(1.3) a little). That is all we need for canonical neighborhood [10] ; see the discussion of II-1.5 in this notes.

Remark 5.5. *However, we have some points with a neck neighborhood, but not a strong neck neighborhood. For example, at time* 0*, the points far away is a tube, but the solution is not defined before* 0 *(by the strong maximum principle). This is not a serious trouble, because if we do surgery, it will automatically become a strong neck; see II-4.4.*

The last inequality,

$$R_{\min} \geq \text{ const } \cdot (1-t)^{-1},$$

for time not close to 1, it follows from Claim 2: this works at the space infinity; for the remaining compact set, there must be *some* constant "const" works for the above.

For time close to 1, we know the standard solution is a cap-like set H attached to a long tube; H is a compact set in the sense of I-11.8; so by I-11.8 it is enough to prove the curvature estimate on the tube part. We argue by contradiction. If the estimate is not true, then for $t_N \to 1$ we have some tubes E_N with

$$R < \frac{1}{N(1-t_N)}.$$

For a *true* tube with $R = 1/N(1-t_N)$, the time it takes to go singular is $N(1-t_N)$. If we start the clock at t_N, at time 1, such a tube has scalar curvature

$$R = \frac{1}{(N-1)(1-t_N)}.$$

So go back to the standard solution. Let $N \to \infty$, we rescale E_N and take limit; like the proof of II-2 Claim 2 we see the limit is the standard tube.

This implies, for sufficiently big N, the standard solution will have some smooth part at time 1, that contradicts to the proof of II-2 Claim 4.

This establishes the curvature estimate.

Remark 5.6. *As an alternative proof, one can use volume comparison because the estimate is true at space infinity.*

Remark 5.7. *The singularity of the standard solution should be of type II, in particular, one cannot get a similar upper bound for $R_{\max}$. But I don't know how to prove it.*

6. Section II-3

Here Perelman gave a clear picture of the manifold at the first singular time, based on II-1.3 through II-1.5.

[9] This is not based on I-11.8 because the curvature is not big enough.

[10] The standard solution can not extend before time 0 because it contains some 0 sectional curvature at $t = 0$. So the head piece of the standard solution at time 0 is not close to a piece of ancient solution if we don't design the head very carefully. But this does not effect later arguments.

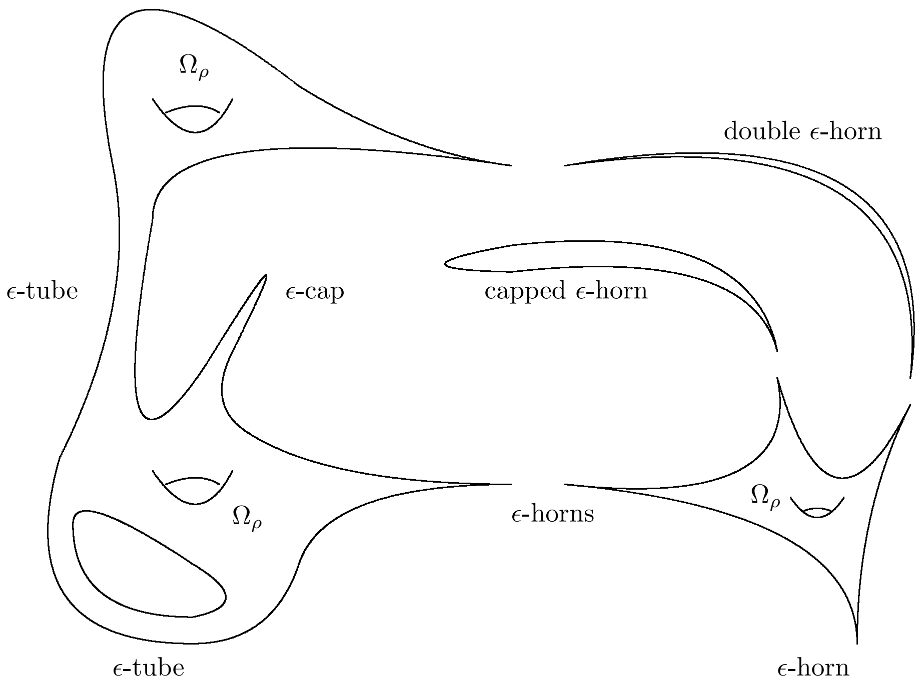
Roughly, pick $\rho \ll r$, where r is the canonical neighborhood parameter, so that there may be a domain Ω_ρ on which curvatures satisfying

$$R \leq \rho^{-2},$$

at the singular time. There could be one or many pieces of Ω_ρ, possibly disconnected, or connected by ϵ-tubes (an ϵ-tube could have both ends on the same Ω_ρ component).

Note Ω_ρ is compact. In fact, by II-(1.3) (see also Claim 1 of II-4.2) and the κ-noncollapsing, every metric ball of radius ρ in Ω_ρ has a lower bound in volume of size ρ^3.

Besides these, there could be fingers (i.e. ϵ-caps or ϵ-horns) that grows out of some Ω_ρ; also, there could be some pieces with high curvature everywhere, i.e. capped ϵ-horns and double horns. [11] At an earlier time, all pieces are connected together with tubes; at the singular time, the middle of some tubes will pinch off, thus generates two horns on each side at the singular time. It is not ruled out that a cap happens to pinch off in the center and generates a single horn, i.e. a degenerate neck pinching. [12]



The argument here is clear in view of II-1.3, II-1.5. One just travels along a long, thin tube to see what happens at the ends, one either reaches a region with relatively low curvature, i.e. Ω_ρ, or, a cap, or a horn where the curvature grows to infinity possibly in finite or infinite distance. One can not get a tube extends to space infinity without forming a horn because the volume then will be infinity.

For example, if the solution is covered by canonical neighborhoods, then it could be diffeomorphic to a space form with positive curvature; otherwise, it contains a tube piece. Trace down the tube as see what we meet, we get either S^3, $\mathbf{RP^3}$, $\mathbf{RP^3}\#\mathbf{RP^3}$ (meet caps on each end), or $S^2 \times S^1$ (meet itself at the ends; note we assume our manifolds are orientable).

[11]It might happen that there are infinitely many double horns.

[12]The Behavior of the standard solution seems to be evidence against the degenerate neck pinching.

We explain why singularities must look like horns. This is because there exist a minimal geodesic γ from regular part to any singularity. Because the gradient estimate II-(1.3) and the canonical neighborhood assumption, sufficiently close to the singularity the curvature R must be big (i.e. no oscillation to low curvature). Then by Claim 2 of I-12.1, we see, after rescale, any points x on γ that is sufficiently close to the singularity lies on the middle of a long minimal geodesic (just the rescale of γ), and by a compactness argument we see the canonical neighborhood of x must be a tube. This tells us, the "path" to a singularity is a tube. We trace γ along this tube toward the singularity and find out that this tube will eventually hit a horn singularity, where R becomes infinity.

I used to think that a horn must have infinite diameter. With nonnegative curvature, using Hamilton's maximum principle, it is true we can not reach infinite curvature within finite distance; but this does not work for arbitrary curvature. In fact one can see the very recent example of neck pinching worked out by Angenent and Knopf.

7. Section II-4.1

Having a clear picture of what is happening at singular time, the basic idea is, we do surgery to round off the tips of horns, and continue running the Ricci flow.

It was not explain in II-4.1 how to do surgery. The section starts with an abstract notion of "Ricci flow with surgery":

Definition 7.1. *A Ricci flow solution with surgery runs on $[0,T]$, with $0 = t_0 < t_1 < t_2 < ... < t_m < T$; on $[t_{k-1}, t_k)$ the soliton is a solution $M_{k-1}(t), t \in [T_{k-1}, t_k)$ of the Ricci flow in the usual sense, and this solution converges to a singular manifold M_{k-1}^{sing} when $t \to t_k$. There are some subsets $\Omega_{k-1}^1, ..., \Omega_{k-1}^{N(k)} \subset M_{k-1}^{sing}$, each of which is a smooth manifold with boundary. The Ricci flow solution with surgery at the moment t_k is a compact, not necessarily connected manifold $M_k(t_k)$; the disjoint union of $\Omega_{k-1}^1, ..., \Omega_{k-1}^{N(k)}$ is isometricly embedded into $M_k(t_k)$.*

We require the Ricci flow solution with surgery satisfies the following a priori *assumptions:*

1). **Curvature pinching:** *there exist a decreasing function $\phi > 0$ with $\phi \to 0$, as $x \to \infty$, and*

$$Rm \geq -\phi(R(t+1))R.$$

2). **Canonical neighbor assumption**. *There exist $r(t) > 0$ depends on time t only, so that whenever at some point (p,t) $R > r^{-2}$, there is a neighborhood around (p,t) as described in II-1.5.* [13]

Remark 7.2. . *We keep in mind, when $t \to t_k$ the solution gets singular; horn singularities will develop. We will remove the tips of the horns at time t_k; the remaining (good) parts are the sets $\Omega_{k-1}^1, ...$ When we glue in the added caps (see II-4.4) we get the manifold $M(t_k)$.*

Remark 7.3. *We will define surgery later. But at this moment, we should keep in mind that surgeries happen at regions with very high curvature, roughly $h^{-2} \gg r^{-2}$, where r is the canonical neighborhood parameter. Also surgeries add a cap-like part (the standard solution) to the solution. So whenever we can control curvature by a reasonable bound we can rule out surgeries; if we can rule out cap type canonical neighborhood we can also rule out surgery for some time before; see for example II-4.3. There will be many other examples later.*

Remark 7.4. . *Clearly the topology could change at the moments t_k. Also, the two* a priori *assumptions hold for solutions without surgeries, i.e. the solution to Ricci flow in the usual sense. 1) is proved in Hamilton's nonsingular paper* [12]*; 2) is proved in I-12.1.*

[13] See the discussion of I-1.5 in this notes.

Remark 7.5. *The pinching estimate above is in fact copied from II-(5.1), here $R(t+1)$ means "R times $(t+1)$". In* [12], *it is proved that*

$$R \geq X(\log X + \log(t+1) - 3) = X \log\left(e^{-3}(t+1)X\right),$$

whenever $X > 0$; here $-X < 0$ is the smallest sectional curvature; we need $X(0) \leq 1$. We explain two versions of II-(5.1) from Hamilton's theorem. Below ϕ means (could be different) function(s) with $\lim_{x\to\infty} \phi(x) = 0$.

First we get a simple version without the time factor: $R \geq X \log(e^{-3}X)$; that implies if R is big, then X, the negative part of Rm is small compare with R; we have $Rm \geq -\phi(R)R$ for $R \gg 0$, where $e^3 e^{1/\phi}$ is the inverse function of $y = x \log e^{-3}x$. So $\lim_{x\to\infty} \phi(x) = 0$. *This is the more classical version of pinching estimate.*

Next, assume $E = R(t+1)$ is big. Then if R big, we can use the previous case to show ϕ is small. Otherwise, $(t+1) = E/R$ is big. If $\log((t+1)X) > N$, *then $-X \geq -N^{-1}R$, so $Rm \geq -N^{-1}R$. Otherwise, we also have*

$$-X \geq -e^N R/E > -N^{-1}R,$$

if E is big enough (depends on N). Thus

$$Rm > -\phi((t+1)R)R,$$

if $(t+1)R$ is big; ϕ is some (other) function with $\lim_{x\to\infty} \phi(x) = 0$.

In Sections II-4 and II-5 Perelman proved the existence of such solution with suitably rescaled (so that $R > -1$, etc. See II-5.) initial metric at $t = 0$. The precise construction of the surgery is given in Sec 4.4, and there the pinching condition 1) will be proved as long as the solution runs.

The canonical neighborhood assumption 2) has strong implications which actually help us to construct the solution with surgery; among other things, we get κ-noncollapsing. So the strategy is, assume $\bar{t}$ is the first time that 2) fails, we extract a converging subsequence as before and conclude that the limit actually satisfies 2), which leads to contradiction. Note 2) holds on $[0, \bar{t}]$, so we have all the desired properties on this time period and so we can continue running the solution.

8. Section II-4.2

One gets some implications of the canonical neighborhood assumption.

Claim 1.

In fact, whenever the curvature is big, i.e. $R(x_0, t_0) > r^{-2}$, we have a canonical neighborhood of (x_0, t_0), on which there is a gradient estimate

$$|\nabla R| < \eta R^{3/2}, \quad |R_t| < \eta R^2,$$

so in this neighborhood R can not grow to more than $Q = R(x_0, t_0) + r^{-2}$.

Remark 8.1. *Note this setting (put r^{-2} in Q) also covers the case that $R(x_0, t_0)$ is not so big.*

The second gradient estimate, $|R_t| \leq \eta R^2$ follows from the estimate of $\nabla\nabla R$, together with

$$R'_t = \Delta R + 2|\operatorname{Ric}|^2.$$

Claim 1, as stated, works for only smooth solutions. We will see, by the construction of surgeries, it is possible to make sense even with surgery, although there we don't know how to define parabolic neighborhoods, etc.

Claim 2.

The proof is basically the same as the proof of the Claim 2 in I-12.1. As before we get a piece of a non-flat cone as a limit. The only problem is, due to the possible presence of a surgery, we don't know if we can extend the time back a little and thus get a contradiction to the strong maximum principle; see [9], or [14] on I-11.4.

However, because the singularities are all of horn type as we have seen before, the canonical neighborhood can not be cap, or any compact manifolds; it can only be a tube (a tube can be close to a very sharp cone). But for a tube, the canonical neighborhood assumption automatically provide some running time back.

Remark 8.2. *Claim 2 does not imply the we can not have infinite curvature within finite distance on an arbitrary (singular) solution. However, this works if we rescaled curvature, especially when we have nonnegative curvature.*

We note, the number Q depends on A.

9. II-4.3; Lemma II-4.3

We assume that the canonical neighborhood assumption holds.

This section verifies that, on an ϵ-horn, the very thin part, i.e. with radius h with $0 \leq h \leq \delta\rho \leq \delta^2 r$, is actually a δ-strong tube. Here notice, *a priori*, we only know it is an ϵ-strong tube. It is significant that δ can be arbitrarily small.

Here we require that the other end of the horn connects to Ω_ρ. Then we know there is a very long distance (in its own scale) from the other end of the horn to our thin part with radius h. This is necessary; otherwise our thin part can be the cap part of a capped horn, and the canonical neighborhood is not a tube.

The problem here is, the presence of surgery makes it not so straightforward to extend time back. A direct application of I-12.1 requires that the solution has been existing for a while, without surgeries, see the statement there; then h will depend on how long the solution has been existing, for which we have no knowledge.

Proof of Lemma II-4.3. The proof starts from the usual compactness-rescale argument. If the conclusion fails, we take a limit X for the counterexamples. Here we need Claim 2 for the existence of limit; it somehow pushes the high curvature regions (thus the "ends") to infinity. Claim 2 does not show that the limit has bounded curvature; that is supplied by splitting, see the next paragraph.

X has two ends because we know the topology, X has nonnegative curvature by the pinching estimate (see Remark 7.5 in this notes), so X splits; so everywhere on X the canonical neighborhood is of strong tube type, and thus we can extend time back.

By the strong maximum principle, the splitting is preserved; so at any earlier time the canonical neighborhood type is again of strong tube type. Here the ϵ-closeness, tells us at the earliest time given by the canonical neighborhood assumption, the curvature is still big enough to guarantee the existence of the canonical neighborhood.

Keep on doing this, we see that X is indeed an ancient solution: we know it splits all the time; in fact, within any finite time interval $[-A, 0]$, each time we move back in time, we get a canonical neighborhood; so the curvature decreases like an ancient solution; the error just add up in a controlled way, i.e. always being ϵ-close to tube. But since X is the rescaled limit, we can do this for ever.

So the limit is an ancient solution with bounded curvature; since we have the canonical neighborhood of tube type all the time, we get some κ-noncollapsing condition. I-11.3 implies X is just

$S^2 \times \mathbf{R}$, so we reached a contradiction, because at $t = 0$ the solution is not just δ-close to a tube, it is exactly a tube. □

10. II-4.4: THE SURGERY

We make more precise the construction of the standard solution. Start from the standard metric on $S^2 \times \mathbf{R}$, denote by z the coordinate on $\mathbf{R}$ direction. We put a conformal factor $e^{-2f(z)}$ on the metric, where

$$f(z) = \begin{cases} 0, & \text{if } z \le 0, \\ e^{-p/z}, & \text{if } z > 0 \text{ and z small .} \end{cases}$$

$p > 0$ is a real number. We only care for small z; for bigger z we can use a warped product metric to close the tube and make a cap, on that part we can guarantee positive sectional curvature.

Remember, if we take $\tilde{g} = e^{-2f} g$, we see

$$\tilde{R}m = e^{-2f} Rm + e^{-2f}(df \circ df) \odot g - \frac{1}{2}|df|^2 e^{-2f} g \odot g + e^{-2f} \operatorname{Hess}_f \odot g,$$

where $\odot$ is the Kulkarni-Nomizu product; see Besse's book. Precisely,

$$\begin{aligned}\tilde{R}_{ijkl} =& e^{-2f} R_{ijkl} + e^{-2f}(f_j f_k g_{il} - f_j f_l g_{ik} - f_i f_k g_{jl} + f_i f_l g_{jk}) + |df|^2 e^{-2f}(g_{ik} g_{jl} - g_{jk} g_{il}) \\ &- e^{-2f}(\operatorname{Hess}_f(i,k) g_{jl} - \operatorname{Hess}_f(j,k) g_{il} - g_{jk} \operatorname{Hess}_f(i,l) + g_{ik} \operatorname{Hess}_f(j,l)),\end{aligned}$$

Now if $\partial_i, \partial_j, ...$ is orthonormal under g (at one point), then $e^f \partial_i, e^f \partial_j, ...$ is orthonormal under $\tilde{g}$. So the sectional curvature is

$$\begin{aligned}\tilde{K}_{ij} =& e^{2f} K_{ij} + e^{2f}(f_j f_j + f_i f_i) - |df|^2 e^{2f} + e^{2f}(\operatorname{Hess}_f(j,j) + \operatorname{Hess}_f(i,i)) \\ =& e^{2f} K_{ij} - f_k f_k e^{2f} + e^{2f}(\operatorname{Hess}_f(j,j) + \operatorname{Hess}_f(i,i))\end{aligned}$$

In our tube case, first we observe that $|f|$, $|df|$, $|\operatorname{Hess}_f|$ are all very small when z is small. So now if K_{ij} is not small, i.e. the i,j-plane is not too close to the $\mathbf{R}$ direction, then $\tilde{K}_{ij}$ is positive. If the sectional curvature of a plane σ is small, then σ must contain a line close to the z direction; say i is that direction; so j, k lies almost in the S^2 direction. We compute

$$\begin{aligned}\tilde{K}_{ij} =& e^{2f} K_{ij} - f_k f_k e^{2f} + e^{2f}(\operatorname{Hess}_f(j,j) + \operatorname{Hess}_f(i,i)) \\ \approx& e^{2f} K_{ij} - \epsilon^2 e^{-p/z} \frac{p^2}{z^4} e^{2f} + e^{2f} e^{-p/z}\left(\frac{p^2}{z^4} - \frac{2p}{z^3}\right) > K_{ij}.\end{aligned}$$

Here repeated index does *not* mean summation; in particular, k means the third direction, pointing roughly in the S^2 direction.

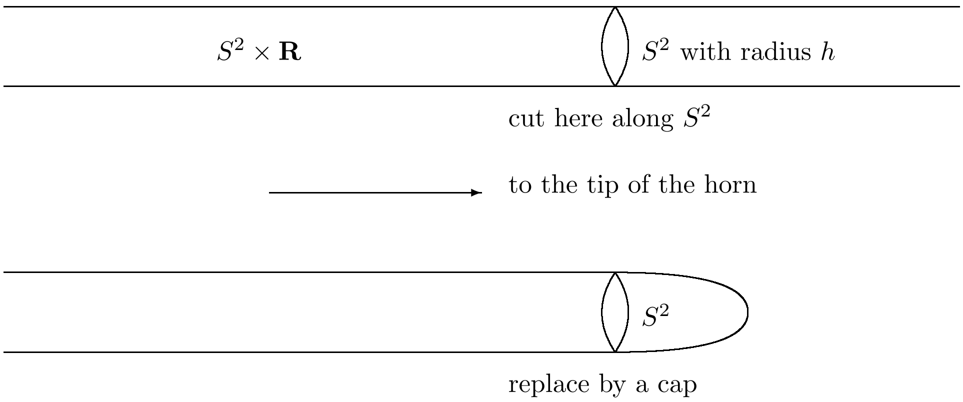
This means we have positive curvature as long as, say, $z \le 10^{-32}$. For big z, we can do a warp product instead of conformal deform to close off the tube to make it a cap with nonnegative curvature.

Then we notice the change in scalar curvature (in 3-dimension):

$$\begin{aligned}\tilde{R} =& e^{2f} R - (n-1)(n-2) e^{2f} |df|^2 - (2 - 2n) e^{2f} \Delta f \\ =& e^{2f} R - 2e^{2f} |df|^2 + 4e^{2f} \Delta f.\end{aligned}$$

So as above, we see R increases.

Now we can carry this argument to δ-tubes, i.e. manifolds that is δ-close to standard $S^2 \times \mathbf{R}$ in $C^{\delta^{-1}}$ norm: we construct the surgery. First we rescale the metric so that the radius of S^2 in the δ-tube has radius about 1. Then We just transplant z to the δ-tube and do the conformal change of metric (and warp product type deform for bigger z). In this way we cut the tube and pasted a cap; and overall we get something $C^{\delta^{-1}}$ close to the standard solution.



Note, the metric is very close to that of $S^2 \times \mathbf{R}$; so only *relative* size of $|df|^2$, Δf, $|\operatorname{Hess}_f|$ will change slightly; in particular, for small z. So the above inequalities implies for small z, R increases, and the minimum of sectional curvatures at each point increases. For not so small z, we actually get positive curvature if δ is small enough.

Also, since we do surgery at places with huge R, we see *the minimum of scalar curvature is not changed.*

In conclusion, the pinching estimate (see Hamilton's nonsingular solution paper [12])

$$R \geq X(\log X + \log(1+t) - 3)$$

is *preserved*; in fact it even improved a little at the surgery regions. The proof in [12] works here without any change, because surgeries only effect regions with high scalar curvature.

11. Definition of surgery

So now we state the definition of surgery precisely:

We run the Ricci flow until a singular time, then there will be a finite number of horns with the other end connect to Ω_ρ, the low curvature region; see II-3.

Besides these, there could be capped horns and double horns, we regard such pieces to be **extinct** and throw them away immediately (but remember the topological information, e.g. connect sums were broken...).

Go back to the remaining horns. Because these horns connect to Ω_ρ, by II-4.3 we find a **surgery radius** h with $0 \leq h \leq \delta\rho \leq \delta^2 r$ and part of the horn (probably close to the singularity in naked eyes) that is δ-close to a tube at scale h.

We rescale this part by h^{-2} (so that we get a tube with radius roughly 1) and apply the surgery described in the previous section of this notes. Then recover the original scale.

After the surgery, we immediately throw away any pieces that is has $R > r^{-2}$ *everywhere* (so these pieces are covered by canonical neighborhoods, the possibilities are: $S^2 \times S^1$, $\mathbf{RP}^3 \# \mathbf{RP}^3$ and spaces diffeomorphic to some space forms with positive curvature, by II-1.5, II-3). We also throw away all pieces that are diffeomorphic to space forms with positive curvature even if the curvature on these pieces are not high somewhere. [14] We then declare such pieces to **extinct**. (But we remember the topological information...)

So we continue to run the Ricci flow until the next singular time and do the same thing...

As long as the canonical neighborhood assumption holds, we see the surgery times are discrete: since the lower bound $R \geq -6$ is preserved, the volume growth is controlled; but at each surgery time, the amount of volume lost is at least h^3.

Exercise 3. *Find out why Hamilton used constant mean curvature foliation to work in the four dimensional case. Why such a foliation exist and unique?*

Part of the answer: See the notes of [12] in the recent collected paper in Ricci flow volume, [1].

12. Lemma II-4.5

Roughly, Lemma II-4.5 says that after a surgery at time T_0, the added cap (we cut the horn and glue in a cap that is almost a standard solution) will remain close to the standard solution for a definite amount of time, on its own scale, with the possibility that somewhere in the direction of the boundary of the cap pinched off at a time very shortly after T_0, so that the cap "extinct", i.e. we just throw the resulting capped horn away.

In this section, the "center" of a cap means the "tip" of a cap; in the standard solution case that means the pole.

Exercise 4. *Will the solution be smooth, if there is a surgery somewhere far away?*

Answer: Yes.

Lemma 12.1. *For all $\theta < 1$, there exists A, so that if the Ricci flow exist on $[0, \theta]$, and there is an upper bound K in curvature over $B_{2A}(p)$; also assume at time 0, $B_{2A}(p)$ is close to the standard solution. Then for all time $0 \leq t < \theta$, $B_A(p)$ is ϵ-close to the standard solution.*

Proof. [15] The proof uses a compactness argument. We make $A \to \infty$ and try to get a limit solution, and show that the limit is indeed the standard solution.

The κ-noncollapsing is missing, but each time we extend time forward a little (depends on K), so that the Ricci flow does not distort distance and volume too much because we have curvature bounds. So we can do a compactness argument and get a limit on this (small) time interval. On the limit, the initial data is exactly the standard solution at 0. By the uniqueness of the standard solution, on this small time interval the limit is the standard solution. In particular, we have a better, explicit curvature control.

In particular we can continue extend time forward until we reach θ. That proves the lemma. □

In order to apply this to prove Lemma II-4.5, we see we need δ to be very small, depends also on A. Only this can guarantee the condition of the above lemma is satisfied.

Remark 12.2. *A priori, we don't know how long the solution exist on the added cap (i.e. would there be singularity/surgery?).*

[14] This is just for convenience. We don't need to do this, in fact in [17] these pieces stays, in order to prove extinction in finite time for certain solutions. Throw away these pieces is more of an *artificial* extinction than a *natural* extinction.

[15] After reading II-2, I realized that the assumption on curvature bound here is essential. We don't need an explicit bound but there must be *some* curvature bound on the whole time interval; otherwise we don't know if the standard solution is unique. I believe this is the basic reason that in the proof of II-4.5 one has to go forward in steps of very small time interval.

Now we apply the above in the proof of Lemma II-4.5. First, the curvature bound is missing; but we get it from the gradient estimate: $|R'_t| \leq \eta R^2$, whenever R is bigger than some r^{-2}, which is roughly the case here. This is implied by the canonical neighborhood assumption. So the curvature can not suddenly grow very big.

Let Q be the maximal scalar curvature on the standard solution on time interval $[0, \theta]$. So at time T_0, the maximal curvature on the cap is at most Qh^{-2} (that bound is quite generous). By the gradient estimate,

$$R \leq \eta^{-1}\Big(\frac{1}{\eta^{-1}Q^{-1}h^2 - t}\Big).$$

So we extend time forward by $\Delta t \leq \epsilon\eta^{-1}Q^{-1}h^2$ and get curvature control, then apply the lemma to conclude that this part of solution is close to a standard solution. In particular, on this part of solution, we get a better curvature control: we still have $R \leq Qh^{-2}$.

Then we repeat this argument. Each time we can extend forward the same time amount $\Delta t \leq \epsilon\eta^{-1}Q^{-1}h^2$. Apply Lemma 12.1, each time the curvature turns out to be better than the gradient estimate could give; after a definite number of steps, we reach $h^2\theta$.

The second part of Lemma 4.5 says, if the solution disappears at some time before T_1, at one point, it must disappear everywhere on the cap. There is only one possibility: the solution pinched off somewhere and the cap is a part of a capped horn. This is because a cap will keep on looking like a cap as long as the solution exits, i.e. no singularity will develop here; the part of a cap with distance Ah to its center does not own a neck that is long enough for surgery.

13. Corollary II-4.6

This corollary says, roughly, if a curve parameterized by $t \in [T_0, T_\gamma]$ in an added cap is long enough (until reach a space or time obstruction), then

$$\int_{T_0}^{T_\gamma} (R(\gamma(t), t) + |\dot\gamma(t)|^2)dt \geq l.$$

This is a technical result; it will be used later to deal with the κ-noncollapsing issue. Note the expression is similar to the L-functional in I-7.

Exercise 5. *Here R should be R_+?*

Answer: No. In the cap, R is big by Lemma II-4.5.

Proof of Corollary 4.6. The notation is, the curve γ is defined on $[T_0, T_\gamma]$.

The first case is, $T_\gamma = T_1 < T$. In particular, $T_1 < T$ means $T_1 = T_0 + \theta h^2$. Then

$$\int_{T_0}^{T_\gamma} R = \int_{T_0}^{T_0+\theta h^2} R \geq \text{const} \int_0^{\theta h^2} \frac{1}{(h^2 - t)} = \text{const} \int_0^{\theta} \frac{1}{(1-t)} = -\text{const} \cdot \log(1-\theta).$$

We can make it bigger than l by choosing θ close to 1. Here "const" means the constant in the last line of II-2 Claim 5.

The second case, $T_\gamma < T_1 \leq T_0 + \theta h^2$. Assume A is so big that on the standard solution $d(q, \partial B(p, 0, A)) > A/4$ at any time $t \in [0, \theta]$ for $q \in B(p, 0, A/2)$. Then

$$\int_{T_0}^{T_\gamma} |\dot\gamma|^2 \geq \frac{1}{T_\gamma - T_0}\Big(\int_{T_0}^{T_\gamma} |\dot\gamma|\Big)^2 \geq \frac{1}{\theta h^2}(\frac{Ah}{4})^2 = A^2/(16\theta).$$

Here of course the distance is distorted (when we compute $|\dot\gamma|^2$), when time changes; but since the solution on the cap is close to the standard solution, and distance shrinks on a standard solution, so we can bound this distortion with a definite factor. In this second case, we only need A big, we don't need θ close to 1. $\square$

14. Corollary II-4.7

The curvature of an added cap will grow like on the standard solution; it may reach infinity at time "1". So if we have control of curvature on a cap, we know the time can not be too long since the surgery.

The condition $R(x, T_x) \leq Q(T_x - T_0)^{-1}$ says, if we rescale the time $T_x - T_0$ to about 1, then $R(x, t_x)$ is bounded by Q. This is also an upper bound for $T_x - T_0$, but it is not a very precise bound; we have in mind on this cap R should be at least h^{-2}. The conclusion of II-4.7 gives a much better bound.

The other inequality, $Q^{-1}R(x,t) \leq R(x, T_x)$, says R at x is not too small at T_x, compare with any other time.

Proof of corollary II-4.7. Argue by contradiction. So we assume that $T_x \geq T_0 + \theta h^2$, that is, the solution exists at least on $[T_0, T_0 + \theta h^2]$.

Notice Lemma 4.5 and Claim 5 of Section II-2 tell us,

$$R(x, T_0 + \theta h^2) \geq \text{const} \cdot \frac{1}{1-\theta} \cdot \frac{1}{h^2},$$

here const is as in II-2 Claim 5. The assumption says $Q^{-1}R(x,t) \leq R(x, T_x) \leq Q(T_x - T_0)^{-1}$; so

$$Q(T_x - T_0)^{-1} \geq R(x, T_x) \geq Q^{-1}\text{const} \cdot \frac{1}{1-\theta} \cdot \frac{1}{h^2},$$

so

$$T_x - T_0 \leq Q^2(\text{const})^{-1}(1-\theta)h^2 < \theta h^2,$$

the last inequality requires that θ is sufficiently close to 1. That gives a contradiction. $\square$

Exercise 6. *What happens to a cap after a while, say near time h^2? Will it go singular everywhere? Or, will it give birth to a "degenerate neck pinching"? Will it happen that the cap behaves like a standard solution for a while up to θ and then the curvature begin to decrease? This means the low curvature region in the boundary direction of a cap will overpower the cap region and eventually smooth the metric.*

Answer: For the standard solution, it goes singular everywhere at $t = 1$; however, it seems to me the head (i.e. center) of the cap is "eaten" by the tube, see the proof of II-1.4. For other cases, the situation is not clear to me.

15. Section II-5.1

The section starts with introducing *normalized* initial metric, that is, $Rm \geq -1$, and each ball of radius 1 has almost Euclidean volume. So the Ricci flow will run at least a definite amount of time before hitting any singularity.

Proposition II-5.1 claims two things for a solution with surgery: first, the solution is κ-noncollapsed, second, it satisfies the canonical neighborhood assumption. However, the parameters κ and r (the critical radius in canonical neighborhood assumption) will decrease with time. The parameter δ (we do surgery at scale $0 \leq h \leq \delta\rho \leq \delta^2 r$) might also decrease in order that the proof gets through.

We recall the following from Perelman's first paper [15]:

Definition. *We say a solution of Ricci flow is* κ**-collapsing** *at* (x_0, t_0) *on scale* $r > 0$*, if* $|Rm| \le r^{-2}$ *at all* (x,t) *with* $d_{t_0}(x, x_0) < r$ *and* $t_0 - r^2 \le t \le t_0$*, and*

$$\mathrm{Vol}_{t_0}(B_r(x_0)) \le \kappa r^n.$$

Lemma. *If a solution of Ricci flow is* κ*-collapsing at* (x_0, r_0) *on scale* r*, then*

$$\tilde{V}(\sqrt[n]{\kappa} \cdot r^2) < 3\sqrt{\kappa}.$$

See I-7.

Remark 15.1. *To be* κ*-noncollapsed is a scale invariant thing, but up to what scale is certainly not. Look at the argument in I-7.3, it does not work for too big a scale* r*; since we have to go back in time by* $\sqrt[n]{\kappa} \cdot r^2$*.*

For any manifold, we can always rescale to get a normalized initial metric. This appears to improve the κ *value in the noncollapsing argument as in I-7.3, but the scale is sacrificed: in order to get normalized initial data, we need to rescale the manifold, as a result, the scale at which we get* κ*-noncollapsing may be small; also, time is getting big. Overall, we did not gain much in this issue; think of the lens spaces.*

16. Lemma II-5.2: κ-noncollapsing

Exercise 7. *Is "* $0 < r < \epsilon^2$ *" in his statement a typo?*

Answer: Probably, yes. But this is not essential. He used ϵ to measure time... So $0 < r^2 < \epsilon$ might be more natural. On page 12, line 10 from top, actually he means $0 < r < \epsilon$. So in the following, I will use $0 < r^2 < \epsilon$.

Exercise 8. *Is* κ *just a constant depending on nothing?*

Answer: Of course when time is big, we get worse control of κ. This is because in the argument of I-7.3, the factor $\tau^{-n/2}$ in the definition of $\tilde{V}$ makes it difficult to get good lower bound of $\tilde{V}$ when τ (inverse time) is big. I-7.3 does work better with ancient solutions, as we have seen in I-11.2; but that is not our situation here.

For another reason that κ gets worse (i.e. smaller) at big time and depends on r, see the last remark in "step 1" below.

The proof of Lemma 5.2 is divided into several steps:

Step 0:

Consider a parabolic neighborhood $P(x_0, t_0, r_0, -r_0^2)$ with $|Rm| \le r_0^{-2}$. It is enough to assume $r_0 > r$, where r is the parameter in the canonical neighborhood assumption. Otherwise, let $r_1 < r$ be the biggest number so that on $P(x_0, t_0, r_1, -r_1^2)$ with $|Rm| \le r_1^{-2}$. There must be a point (x,t) with $t \in [t_0 - r_1^2, t_0]$ a *latest* time so that $|Rm(x,t)| = r_1^{-2}$, $(x,t) \in P(x_0, t_0, r_1, -r_1^2)$. and we want to show κ-noncollapsing at (x_0, t_0).

Now the pinching estimate tells us at (x,t) the scalar curvature R is also big (see Remark 7.5 of this notes), we have canonical neighborhood around (x,t). We see the ball $B(x, t, R(x,t)^{-1/2})$ has a lower bound in volume, of magnitude $C_2^{-2} R^{-3/2}$ (see II-1.5 for this bound; also by II-3, II-4 we throw away components that are diffeomorphic to positive space forms). The actual definition of κ-noncollapsing is only slightly different from this lower bound; but it can be easily proved by the estimate of distance-volume distortion; see [11] Theorem 17.1.

Remark 16.1. *The above is not only helpful, it is also necessary. In fact, the proof below for scale more than r does not work for scale smaller than r. The problem is, at small scale, we need to deal with curvature bigger than r^{-2}, so it is possible we are in the added cap formed just after a singular time; such a region does not exist before the surgery. So it is impossible to define $\mathcal{L}$-geodesics (see II-7) starting from such a region in a reasonable way, in particular, we can not get a good definition for the reduced volume based at these regions.*

On the other hand, the construction of surgery tells us, there is no surgery on a region with $R < r_0^{-2}$. So we can define $\tilde{V}$, etc. and prove that the solution is κ-noncollapsing up to scale ϵ; see the previous section of this notes.

Step 1: The proof of Lemma II-5.3.

In the notation of I-7, the conclusion is

$$\int_0^{t_0-T_0} \sqrt{\tau}\,(R_+(\gamma(\tau),\tau) + |\dot{\gamma}(\tau)|^2)d\tau \geq \mathcal{L}.$$

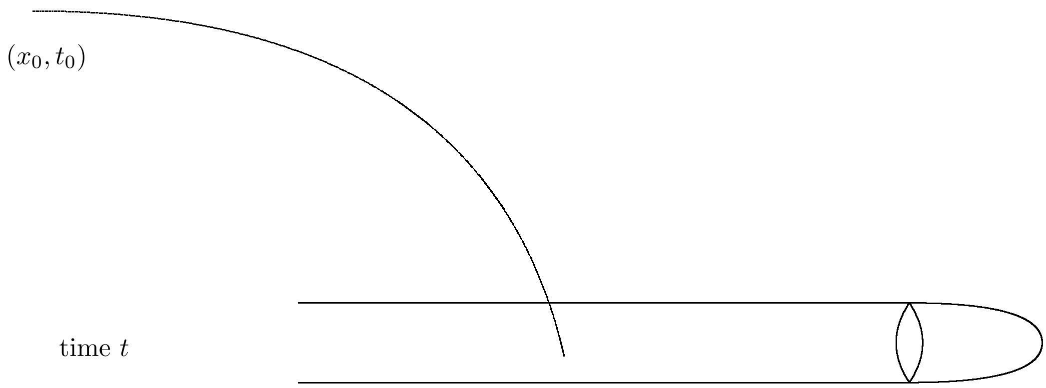
Here $\tau = t_0 - t$ and for convenience we reverse the direction of γ: i.e. $\gamma(0) = x_0$. The readers have no difficulty to recognize the directions t and τ and other notations.

The functional L is a quantity with unit; it measures length. I would prefer to write $\mathcal{L}r_0$ instead of $\mathcal{L}$, so that makes $\mathcal{L}$ scale-invariant. But let's follow the version in [16].

Proof of Lemma 5.3. As before, We start from a neighborhood $P(x_0, t_0, r_0, -r_0^2)$ on which $|Rm| \leq r_0^{-2} < r^{-2}$. Assume γ is a curve in space-time that starts at (x_0, t_0) and ended at some added cap. More precisely, we define a *barely admissible curve* as the following:

Definition. *A* **barely admissible curve** *is a curve defined on $[t, t_0]$, so that t is a surgery time, and $\gamma(t) \in B(p, t, Ah/2)$, where p is the center of the add cap.*

Recall h is the radius of the tube at which the surgery happened. Here is a picture of a barely admissible curve:



Remark 16.2. *A barely admissible curve enters into a region (an added cap) that probably does not exist a second ago. Later we will be doing some differential inequalities and we want to avoid such unstable region. So we want to single out these curves.*

Compute, for some Δt,

$$\int_0^{\Delta t} \sqrt{\tau}|\dot\gamma(\tau)|^2 d\tau \geq \Big(\int_0^{\Delta t} \frac{1}{\sqrt{\tau}} d\tau\Big)^{-1}\Big(\int_0^{\Delta t} |\dot\gamma(\tau)|d\tau\Big)^2.$$

Now we pick $\Delta t = \epsilon r_0^4 \mathcal{L}^{-2}$,

$$\int_0^{\Delta t} \sqrt{\tau}|\dot\gamma(\tau)|^2 d\tau \geq \frac{1}{2\sqrt{\epsilon} r_0^2}\mathcal{L}\Big(\int_0^{\Delta t} |\dot\gamma(\tau)|d\tau\Big)^2.$$

So if $\gamma([0,\Delta t])$ runs out of $P(x_0,t_0,r_0,-\Delta t)$, because of our assumption on P that the curvature $|Rm| \leq r_0^{-2} < r^{-2}$, during this Δt time period, the length of a fixed vector is distorted by a factor at most $e^{-\epsilon r_0^2 \mathcal{L}^{-2}}$, which is very close to 1; see section 17 of [11]. So by estimating the part of the distance in $P(x_0,t_0,r_0,-\Delta t)$, we get a lower bound [16]

$$\int_0^{\Delta t} |\dot\gamma(\tau)|d\tau \geq \sqrt{2}\,\epsilon^{1/4} r_0,$$

and this implies our conclusion

$$\int_0^{t_0-T_0} \sqrt{\tau}(R_+(\gamma(\tau),\tau)+|\dot\gamma(\tau)|^2)d\tau \geq \mathcal{L}.$$

The other case is, $\gamma([0,\Delta t])$ lies in $P(x_0,t_0,r_0,-\Delta t)$. Then let's check the assumption in Corollary II-4.6: we have $T_\gamma < t_0-\Delta t$ (notation: the part of γ that lies in an added cap in the sense of Corollary II-4.6 is defined on $[T_0,T_\gamma]$); this is because we choose δ small so the neighborhood $P(x_0,t_0,r_0,-\Delta t)$ is not the place on which a replaced cap evolves: the curvature bound r_0^{-2} is not big enough. So the curve satisfies the assumptions of Corollary II-4.6. Applying Corollary II-4.6 with $l = \mathcal{L}(\Delta t)^{-1/2}$, we get again

$$\begin{aligned}\int_0^{t_0-T_0} \sqrt{\tau}(R_+(\gamma(\tau),\tau)+|\dot\gamma(\tau)|^2)d\tau \geq& \sqrt{\Delta t}\int_{T_0}^{T_\gamma}(R_+(\gamma(t),t)+|\dot\gamma(t)|^2)dt\\ \geq& l\sqrt{\Delta t} = \mathcal{L}.\end{aligned}$$

Here $t = t_0-\tau$; that is good with the notation of Corollary II-4.6. □

Remark 16.3. *The use of Corollary 4.6 requires that δ is sufficiently small. Certainly it depends on $\mathcal{L}$.*

Remark 16.4. *A barely admissible curve is remarkable in the sense that it goes deeply into a cap (or a horn, a split ago). So we expect its l value to be big.*

Remark 16.5. *We want the barely admissible curves have large l value. The reason we want this is, we want to run differential inequality about the minimum of l; so we don't want the minimum happens at a region that does not exist a second ago. See also the discussion of II-6.3(a) in this notes.*

One has to divide L by $2\sqrt{\tau}$ to get l; this means, if a curve is long in the sense of time, big L does not automatically imply big l. This is not a serious problem, but enough to hint that the parameters like δ should not stay constant all the time, we need better parameters to get better estimates.

A more serious problems happens in the following situation: assume a curve runs in space-time from (x_0,t_0) back to a time close to a point (x,t) with small time, say, $t \ll 2^{i-N}\epsilon$, and (x,t) lies in an added cap; then Lemma 5.3 does not give the desired lower bound. In fact Lemma 5.3 need a certain small δ, that is usually not the δ (coupled with a r value) value which is already constructed at t, and for which **we have no right to change***: we are arguing by an induction.*

We remark, though, if the time period is not too close to 0, say if $i \gg 5$, and $t \sim 2^{i-k}\epsilon$, $k = 0,...,5$, δ values are small enough to get big l value. This will be used in the next step.

[16] This is generous. Probably take $\Delta t = r_0^4\mathcal{L}^{-2}$ will be sufficient for the argument.

For this reason, we can get $\min l \leq 3/2$ *only on, say,* $[2^{i-5}\epsilon, 2^i\epsilon]$. *Can not extend time back more, because then we don't know how to prove that barely admissible curves have big* l *value and how to carry out the differential inequality argument in I-7.*

Step 2: We first supply the proof of $\min l \leq 3/2$ on $[2^{i-5}\epsilon, 2^i\epsilon]$. We take $\mathcal{L} = 2\epsilon^{-2}T\sqrt{T}$ (we want to show some κ-noncollapsing on $[0, T]$, note the canonical neighborhood assumption works on this time interval); so Lemma 5.3 says bare admissible curves have $L_+ \geq 2\epsilon^{-2}T\sqrt{T}$; here L_+ is to replace the R in L by R_+. but

$$L_+ < L + 4T\sqrt{T}, \tag{16.6}$$

because $R > -6$; so for barely admissible curves we still have $L \geq \epsilon^{-2}T\sqrt{T}$. So we can now apply the differential inequality argument in I-7 because barely admissible curves do not carry minimum. So we conclude $\min l \leq 3/2$, see I-7.1.

Next, we want to get a curve with small l value, and also with small curvature at the end: only then can we bound l nearby.

We claim there exist a point (x, t) with $t \in [2^{i-1}\epsilon, 2^i\epsilon]$, $R(x, t) < r_{i-1}^{-2}$, and $L_+ \leq \epsilon^{-2}T\sqrt{T}$. [17]

Note $2^i\epsilon \leq t_0 \leq 2^{i+1}\epsilon$, so $T \sim 2^i\epsilon$.

If the claim is not true, as in the remarks of the previous step, we have a curve from (x_0, t_0) to some $(x, 2^{i-1}\epsilon)$ with $l \leq 3/2$; but the curvature on the end is big. We can assume $R(\gamma(\tau), \tau) \geq r_{i-1}^{-2}$ for all $\tau \geq 2^{i-1}\epsilon$, i.e. time stay away from t_0 and deep into $[2^{i-1}\epsilon, 2^i\epsilon]$; otherwise we can take part of this curve and get both small l and small R in the end. So now we estimate

$$L \geq \int_\gamma \sqrt{\tau}R(\gamma(\tau), \tau) \geq \int_0^{2^{i-1}\epsilon} (\sqrt{\tau}) r_{i-1}^{-2} - 6\int_0^{2^i\epsilon} (\sqrt{\tau}) > \frac{1}{3}(2^{i-1}\epsilon)^{\frac{3}{2}} r_{i-1}^{-2}.$$

Here we used $R \geq -6$. Thus $l \geq 2^{i-1}\epsilon r_{i-1}^{-2}/3$ since time$\sim 2^{i-1}\epsilon$. Because $r_{i-1}^2 < \epsilon$ (sure!), that contradicts to $l \leq 3/2$.

Remark 16.7. *Here* ϵ *(a constant measure closeness in the definition of canonical neighborhood assumption, I-1.5) is used here to measure time. It is not clear to me why he used it for this purpose here. Apparently other numbers also works.*

Step 3:

We follow the previous proof of κ-noncollapsing as in I-7.3. This is like Gromov's generalization of the volume comparison beyond cut locus: when L-geodesic hits surgery region, we just stop. As a result, l is defined only at those points that can be hit by a minimal geodesic that runs away from surgery regions; elsewhere we take l to be $+\infty$.

Let $\tilde{V}$ be defined as before, so it is again a decreasing function of τ.

We use the curve in the previous step. Note the end (x, t) has $R \leq r_{i-1}^{-2}$, so by Claim 1 of Section II-4.2, we get an upper bound in curvature on a parabolic neighborhood of (x, t) with definite size. We have upper bound of l at (x, t), so we can get a lower bound of $\tilde{V}$ at a slightly earlier time than t, as in I-7.3. Thus by monotonicity we proved the κ_i-noncollapsing.

□

Remark 16.8. *Look at I-7.3, we see our argument here shows* κ_i*-noncollapsing up to scale* $\sqrt{\epsilon}$.

[17] This is generous. $L\sqrt{L}$ is used to deal with the relation between L_+ and L; here I think, $10T$ is sufficient.

17. II-5.4: Canonical neighborhood assumption

Proof of Proposition 5.1. We argue by contradiction. By induction, we assume there is a sequence of solutions, $M^{\alpha,\beta}$, such that $r^{\alpha\beta} \to 0$, $\delta^{\alpha\beta} \to 0$ on $[2^i\epsilon, 2^{i+1}\epsilon]$, but there are counter examples to the canonical neighborhood assumption during the time interval $[2^i\epsilon, 2^{i+1}\epsilon]$, with canonical neighborhood parameter $r^{\alpha,\beta}$, and the surgery parameter $\delta^{\alpha,\beta}$. For convenience, we omit the superscript α, β.

Let $\bar{t}$ be the first time that the canonical neighborhood assumption is violated, say at $\bar{x}$. This is a little subtle. We may want to redefine the canonical neighborhood condition by replacing ϵ there by $(1-2^{-i})\epsilon$ on time interval $[2^i\epsilon, 2^{i+1}\epsilon]$. So roughly, at this first time of violation, the neighborhood of $(\bar{x}, \bar{t})$ is at most *exactly* $(1-2^{-i-1})\epsilon$ close to a corresponding set of some ancient solution; can not be any closer. Our strategy here is, by taking limit we will show it is actually much closer to the corresponding set of an ancient solution.

By definition, $R(\bar{x}, \bar{t}) > r^{-2}$.

So up to time $\bar{t}$, the canonical neighborhood assumption is satisfied. In particular, we can use all the results from Section II-4, II-5 up to time $\bar{t}$. In particular, we have the noncollapsing condition ; after a look at the proof of Lemma 5.2, especially step 2 and 3 in this notes, we see even though $r^{\alpha\beta} \to 0$, $\delta^{\alpha\beta} \to 0$, we get a κ-noncollapsing condition independent of $r^{\alpha\beta}, \delta^{\alpha\beta}$; κ depends on the parameter r, on the previous time interval. That allows us to take limit.

Now we rescale the Ricci flow around $(\bar{x}, \bar{t})$ by factor $R(\bar{x}, \bar{t})$. Claim 2 of II-4.2 tells us we can bound curvature within a definite distance A. So we can take limit at least on this part. However, since $r^{\alpha\beta} \to 0$, $\phi(R(\bar{x}, \bar{t})) \to 0$. So in fact we can take bigger and bigger A in Claim 2 of II-4.2. So we get a completed manifold $M^\infty(0)$ with nonnegative curvature as the limit. Now by the canonical neighborhood assumption, we can argue like in I-11, and conclude the limit has bounded curvature, say $|Rm| \leq Q_0$.

At this stage of the proof, we don't know yet if this solution at $t = 0$ can be extended back in time, we don't even know that the "surgery times" on limit are discrete. So in the following we (implicitly) consider a solution in the sequence $M^{\alpha,\beta}$ that is *sufficiently* close to the limit; then argue like II-4.4 we know on this solution the surgery time is discrete. With the help of this, we eventually will show the solution on the limit can be extended back in time and there is indeed no surgery on the limit; i.e. for sufficiently big α, β, there is a large parabolic neighborhood around $\bar{x}^{\alpha,\beta} \in M^{\alpha,\beta}$ that has no surgery.

Now we remind the readers Claim 1 of II-4.2 says, *before rescale*, in our situation, if $R(x,t) \geq r^{-2}$ then $R \leq 8R(x,t)$ over the parabolic neighborhood

$$P\Big(x, t, \frac{1}{2}\eta^{-1}R(x,t)^{-1/2}, -\frac{1}{8}\eta^{-1}R(x,t)^{-1}\Big),$$

by Claim 1 of II-4.2; here η is the gradient estimate constant appeared in II-(1.3).

Without surgeries, the above tells us that *after rescale*, we have curvature control with time extend back for $\epsilon\eta^{-1}Q_0^{-1}$. That means we can actually extend back in time and make the limit a solution $M^\infty(t)$, $t \in [-\epsilon\eta^{-1}Q_0^{-1}, 0]$.

Let's first assume this is not the case. So *before rescale*, there is a point y, and a time $T_0 \in [\bar{t} - \epsilon\eta^{-1}Q_0^{-1}R^{-1}(\bar{x}, \bar{t}), \bar{t}]$, so that at y the Ricci flow is defined at T_0 but not before (i.e. y is on an add cap); moreover, $d^2_{T_0}(\bar{x}, y)R(\bar{x}, \bar{t})$ stay bounded in order that x does not escape to infinity when taking limit. We assume T_0 is the first such time before $\bar{t}$ (here we mean on $M^{\alpha,\beta}$ with big α, β).

Even with surgery, we still have curvature control at $\bar{x}$ at time T_0 by II-4.2 Claim 1 (assume Q_0 is bigger than $\epsilon^{-1}\eta$). Then we can use Claim 2 in II-4.2 at time T_0 to control the curvature at y, which is $\approx h^{-2}$ since y is on the added cap.

That shows $h^2(T_0)R(\bar{x},\bar{t})$ is bounded away from 0. [18]

So $\bar{x}$ lies very close to a (replaced) cap: $\bar{x}$ is within distance, say, Ah to the tip of the cap (almost y), at time T_0. Moreover, the curvature is bounded at $\bar{x}$ by (use II-4.2 for the first inequality)

$$8^{-1}R(\bar{x},t) \leq R(\bar{x},\bar{t}) < \epsilon^{-1}\eta Q_0 R(\bar{x},\bar{t}) \leq \frac{1}{\bar{t}-T_0}.$$

So we apply Lemma 4.7 to conclude $\bar{t}-T_0$ is relatively small compare with h^2, $\bar{t}-T_0 \leq \theta h^2$.

By lemma II-4.5, during this time period, near $\bar{x}$ the manifold is close to the standard solution, in particular, on the limit it is much better than $\epsilon/2$-close. By Claim 5 in II-2, we get a contradiction. So we can extend time back and make the limit a solution $M^\infty(t)$, $t \in [-\epsilon\eta^{-1}Q_0^{-1}, 0]$.

We apply the same argument again: as before, we have a curvature bound Q_1 for $M^\infty(-\epsilon\eta^{-1}Q_0^{-1})$. Then we try to extend time back from $t = -\epsilon\eta^{-1}Q_0^{-1}$. We use the same argument to see that we can extend time back again by $\epsilon\eta^{-1}Q_1^{-1}$ with no surgery, because otherwise $(\bar{x},\bar{t})$ would have a canonical neighborhood.

This works. Remember we proved $R(\bar{x},\bar{t})h^{-2}$ is bounded away from 0; but we don't know how far it is from 0, because, *a priori*, there is no knowledge on how big $d_{T_0}(\bar{x},y)$ is. So we don't know how far $\bar{x}$ is from the tip of the cap. This is fine, because $\delta^{\alpha,\beta}, r^{\alpha,\beta} \to 0$ so II-4.7 is valid for arbitrarily big Q. This same principle works when we try to extend the time further back: whenever we hit a surgery at time T_0', we get *some* lower bound of $R(\bar{x},\bar{t})h^{-2}$. By II-4.2 Claims 1, 2, we get

$$(Q^*)^{-1}R(\bar{x},t) \leq R(\bar{x},\bar{t}) \leq \epsilon\eta^{-1}(Q_0^{-1}+Q_1^{-1})\frac{1}{\bar{t}-T_0'},$$

where $\bar{t}-T_0' < R(\bar{x},\bar{t})^{-1}\epsilon\eta^{-1}(Q_0^{-1}+Q_1^{-1})$ is the amount of time we can extend back before hitting a surgery.

The lower bound of $R(\bar{x},\bar{t})h^{-2}$, and the constant Q^* depend on Q_1, Q_0, and where the surgery happens; e.g., the first inequality is obtained by using II-4.2 Claim 1 on consecutive time intervals. We have no knowledge of these constants, only knowing they are uniform for α,β. It is OK to apply II-4.7 because $\delta^{\alpha,\beta}, r^{\alpha,\beta} \to 0$. [19]

So the limit turns out to be a solution $M^\infty(t)$, $t \in [-\epsilon\eta^{-1}(Q_0^{-1}+Q_1^{-1}), 0]$.

We keep on doing this and extend time back. We get an ancient solution unless the curvature bounds Q_i goes to $+\infty$; but that is ruled out by an argument as in I-12.1. We briefly review that argument:

Assume the solution exists on $(t', 0]$, but can not extend before t'. By the Harnack inequality $R_t + R/(t-t') \geq 0$, together with the pinching estimate, we conclude

$$0 \leq R \leq \frac{|t'|}{t-t'}Q_0.$$

Then we estimate that distance will distort by at most $100|t'|\sqrt{Q_0}$; see I-8.3(b). So we will speak of "huge" distance without mentioning time.

First assume the limit is compact. Then $R_{\min}$ is nondecreasing, that implies at any time there is region with low scalar curvature. The gradient estimate tells us near time t' we can extend this region. Now the previous lemma implies that if there exists high curvature region, then it will be pushed to infinity and make the visible region enjoying curvature bounded by $c(\kappa)^{-1}$. In particular, we can extend the solution back.

[18]This is interesting, since $h < \delta\rho < \delta^2 r$ is very small; on the other hand there is no way to control Q_0, so we have to use II-4.2 Claim 2 to argue. (Use II-4.2 Claim 1 one only gets that h^2Q_0 is bounded away from 0).

[19]This argument suggests that a long cap will eventually develop into singularity; in some sense no singularity afterwards implies no cap.

For the noncompact case, we find a distance D at time 0 so that every point z at distance more than D from $\bar{x}$ is close to being a midpoint of a long minimal geodesic; by saying this we mean that starts from z, there are two long minimal geodesics γ_1, γ_2 on $[0, D]$ so that

$$d(\gamma_1(D), \gamma_2(D)) \geq (2 - \epsilon)D.$$

To prove this, we argue by contradiction, assume z_i are counterexamples associated to D_i; $D_i > 2D_{i-1}$. Then by compactness there are $i < j$ so that the angle between the geodesics from the base point $\bar{x}$ to z_i and to z_j are small. Now apply the Toponogov triangle comparison theorem, we see z_i actually is not a counter example. That is a contradiction.

Now assume such a z has a huge curvature near t'. Then it has a tube neighborhood because by the above argument the cap neighborhood is ruled out. The radius, $s(z)$, of this tube at z near t' is extremely thin. At time 0 the topology near z does not change; moreover since $Rm \geq 0$ and the distance distortion is controlled, we conclude, by removing a ball centered at z, the manifold will become disconnected; go to time 0, the size of this ball is modified by only a definite amount while volume decreases. Now we see his can not happen, in view of the volume comparison from ∞; i.e. Yau's "visibility" argument (using this Yau proved that a noncompact manifold with $\mathrm{Ric} \geq 0$ has at least linear volume growth; see [23]).

So $R(z)$ must be uniformly bounded as long as $d(z, \bar{x}) > D$ (at any time). Then by Claim 2 of 4.2 we get global curvature control. Thus we can extend the solution back in time more.

So $(\bar{x}, \bar{t})$ turns out to be lying on an ancient solution and we reached a contradiction to the choice of $(\bar{x}, \bar{t})$. Compare Remark 21.2 of this notes. That proves the conclusion. $\square$

18. II-6.1

If the scalar curvature is nonnegative at some time, it will become positive immediately unless it is Ricci flat; see equation II-(7.1). Remember $R'_{\min} \geq 2R^2_{\min}/3$, it will go extinct within finite time because surgeries does not effect $R_{\min}$.

At each surgery time, we throw away pieces that are capped horns or double horns; after the surgery, we throw away those components that are diffeomorphic to a manifold with constant positive sectional curvature. The remaining parts can experience surgeries later, but eventually $R_{\min}$ will grow bigger than r^{-2} so the solution is covered by canonical neighborhoods. So these parts are either $S^2 \times S^1$, or $RP^3 \# RP^3$; because by definition of extinction (II-3 and II-4.1), the only possible canonical neighborhood types are a) and b) in II-1.5.

As we have seen in II-3, we can recover the initial manifold. This gives the structure theorem of orientable manifolds with positive scalar curvature: any such manifold is a connect sum of spaces forms with positive curvature and some copies of $S^2 \times S^1$. Remember, when we glue in a double horn, it means either connecting two components, or glue a $S^2 \times S^1$ to one component.

The work of Gromov and Lawson tells us that these manifolds do support metrics with positive scalar curvature.

19. II-6.2

This section restates Theorem I-12.2. We will use the version with surgeries, whose proof is given in II-6.3(c) below.

20. Proposition II-6.3

We have in mind, the results in Section 6 deals with "long time behavior". The scale r_0 is usually getting big as time goes by. The (small) number $\bar{r} = \bar{r}(A)$ is a "quantity without dimension", i.e. it does not have a unit: not length, not time.

The main result is Proposition II-6.3:

Proposition. *For $A < \infty$ there is $\kappa > 0$, $K_1(A) < \infty$, $K_2(A) < \infty$, $\bar{r}(A) > 0$ so that: For all $t_0 < \infty$ there exists $\bar{\delta}$, for any solution with $\delta(t)$-cutoff with $\delta(t) \leq \bar{\delta}$ on $[0, T]$, and normalized initial data, if the solution is defined on $P(x_0, t_0, r_0, -r_0^2)$, $2r_0^2 < t_0$, and $|Rm| \leq r_0^{-2}$ on P, $\mathrm{Vol}_{t_0}(B(x_0, t_0, r_0)) \geq A^{-1}r_0^3$, then :*
(a) The solution is κ-noncollapsed on the scale less than r_0 in $B(x_0, t_0, Ar_0)$.
(b) Every point $x \in B(x_0, r_0, Ar_0)$ with $R(x_0, t_0) \geq K_1 r_0^{-2}$ has a canonical neighborhood.
(c) If $r_0 \leq \bar{r}\sqrt{t_0}$, then $R \leq K_2 r_0^{-2}$ in $B(x_0, r_0, Ar_0)$.

By the curvature bound, we can assume the lower bound in volume is valid for all $t \in [t_0 - r_0^2, t_0]$.

Exercise 9. *Why do we need a).? Isn't it already proved in II-5.2?*

Answer: This theorem, under its condition, gives κ-noncollapsing no matter how big the time is. In II-5.2, the κ value gets worse at big time. The theorem here is useful when we take limit with time goes to infinity.

(b) says, every point $x \in B(x_0, r_0, Ar_0)$ with *relatively* big curvature has a canonical neighborhood.

Exercise 10. *What is the application of part (b)?*

Answer: in II-7.2(b).

Exercise 11. *What's the meaning of the assumption $r_0 \leq \bar{r}\sqrt{t_0}$ in (c)?*

Answer: Simply put, this implies we have a good curvature pinching, from [12]. See the proof, and Remark 7.5 in this notes.

Proof of (a). We only need to study those r_0 that are more than $r(t_0)$, the canonical neighborhood parameter. Observe, the conclusion of the proposition says we have κ-noncollapsing at scale up to r_0. When $r_0 < r(t_0)$, this is almost automatic. In fact, pick any point (x, t), find the maximal r_1 so that $|Rm| < r_1^{-2}$ on $P(x, t, r_1, -r_1^2)$. If $r_1 \leq r(t_0)$, we know some points in $P(x, t, r_1, -r_1^2)$ has big curvature and so owns a canonical neighborhood. However, canonical neighborhoods are κ-noncollapsed (we throw away components diffeomorphic to positive space forms so the canonical neighborhoods are caps and tubes). So the solution is κ-noncollapsed at (x, t) at scale r_1, because in $P(x_0, t_0, r_0, -r_0^2)$ volume and distance distort by at most a factor $e^{c(3)}$, so we get a lower bound of $\mathrm{Vol}(B(x, t, r_1))$ as well.

If $r_1 > r(t_0)$, the κ noncollapsing follows from the proposition with the setting $r_0 = r_1 > r(t_0)$; which we will prove below.

Remark 20.1. *It is necessary to distinguish the small scale case from the large scale case. The reason is the same as in the proof of II-5.2. At scale smaller than the canonical neighborhood parameter $r(t_0)$, we may have trouble defining the reduced volume $\tilde{V}$ based at high curvature region.*

So now assume $r_0 > r(t_0)$.

If the proposition is not true, there is a point (x, t) with $d(x, x_0, t_0) = Ar_0$ (assume this, like in I-8) at which the solution is quite collapsed at some scale ρ with $r(t_0) \leq \rho \leq r_0$; note we assume $r(t_0) \leq \rho$ because of the discussion in the beginning of the proof.

Rescale r_0 to 1 and move time t_0 to 1. So the curvature estimate $|Rm| \leq 1$ holds on $P(x_0, 1, 1, -1)$. By using the same notation, $\rho \leq 1$.

We make $p = x$ and define $\tau = 1 - t$, l, $\tilde{V}$, etc. as in I-8. We see from I-7.3,

$$\tilde{V}(\sqrt[n]{\kappa} \cdot \rho^2) < 3\sqrt{\kappa}.$$

That is the reason we can not use this method to show noncollapsing on a bigger scale: we need $\sqrt[n]{\kappa} \cdot \rho^2 < 1/2$ in order to use monotonicity of $\tilde{V}$.

We want to show that $\tilde{V}(1)$ is not so small and thus get a contradiction to the monotonicity of $\tilde{V}$. We claim, it is enough to show that $\min l(y, 1/2)$ is not so big, here the minimum is taken among y with $d_{1/2}(y, x_0) \leq 1/10$.

In fact, $d_{1/2}(y, x_0) \leq 1/10$ implies $d_0(y, x_0) \leq 1$ because the curvature is bounded by 1. Thus if $\min l(y, 1/2)$ is reached at some $y_{1/2}$, we can extend the geodesic in space-time to all points y_1 with $d_0(x_0, y_1) \leq 1$ without increasing l too much, thanks the curvature bound. So we have a uniform bound of l over $B_1(x_0)$ at time $t = 0$, i.e. $\tau = 1$, together with the lower bound in volume (this is the only place it is used), we get a lower bound in $\tilde{V}$.

Now unlike I-8, we have surgeries here, and remember once we reach an added cap region at a surgery time, we stop (like hit a cut point in the usual geodesic case). So we need to show that barely admissible curves has relatively big l value, or equivalently, big $\bar{L}$ value when $\tau \leq 1/2$; in particular, the minimum of l does not happen at the end of a barely admissible curve (these region may not exist *before* the surgery time). Then we can follow the differential inequality argument as in I-8 without any change. This is our basic strategy.

Recall, defined in I-7.1 and used in I-8, $\bar{L} = 2\sqrt{\tau}L$, with $\tau = 1 - t$. Lemma II-5.3 tells us, we can make

$$\bar{L} = 2\sqrt{\tau} \int \sqrt{\tau}(R(\gamma(\tau), \tau) + |\dot{\gamma}(\tau)|^2) d\tau \geq \sqrt{\tau}\, C^*(A) \tag{20.2}$$

for barely admissible curves. This estimate, unfortunately, become less powerful when τ is small, i.e. when there is a surgery at time close to t_0.

We need a better growth estimate of $\bar{L}$, in order to show that barely admissible curves does not contribute to the minimal of $\bar{L}$ near x_0: as the proof of I-8.2, we get a growth estimate for the minimum of some function h (essentially $\min \bar{L}$); if this minimum is obtained by some barely admissible curve, i.e. a curve passing through a cap region, then this minimum may jump because we have to throw away those part of a barely admissible curve after the cap region; then the argument does not make sense.

Lemma 20.3. *$\bar{L} + 2\sqrt{\tau} \geq 0$ for $t \geq 1/2$, i.e. $\tau \leq 1/2$.*

Proof of Lemma. We have $R \geq -3$ (after rescale; it can well be a blow down, when $r_0 > 1$). In fact, by the differential inequality $R'_{\min} \geq 2R^2_{\min}/3$, we estimate

$$R \geq -\frac{3}{2} \cdot \frac{1}{t + 1/4} \cdot r_0^2 \geq -3;$$

The last inequality follows by $r_0^2 < t_0/2$.

So now we estimate

$$L \geq \int_0^\tau \sqrt{s} R(\gamma(1-s), 1-s) ds \geq -3 \int_0^\tau \sqrt{s} ds = -2\tau^{3/2}.$$

So (use $\tau \leq 1/2$ for the last inequality).

$$\bar{L} = 2\sqrt{\tau} L \geq -4\tau^2 \geq -2\sqrt{\tau}.$$

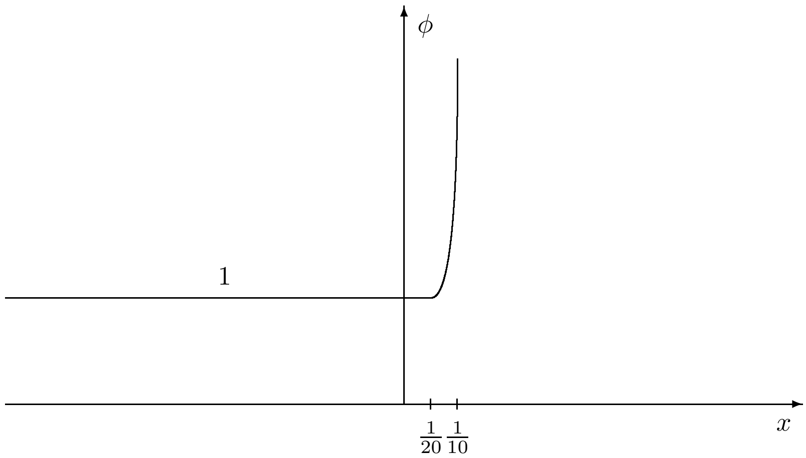
□

Write $d(\cdot, t) = d(x_0, \cdot)$ at time t.

Pick the function ϕ with $\lim_{x\to 1/10^-} \phi(x) = \infty$ as in I-8.2; remember we can make

$$\frac{2(\phi')^2}{\phi} - \phi'' \geq (2A + 300)\phi' - C(A)\phi;$$

for example, let $\phi = (0.1 - x)^{-2}$ near $1/10$, then smooth connect to constant 1 near $1/20$.



We define

$$h(y, t) = \phi(d(x_0, y, t) - A(2t - 1)) \cdot (\bar{L}(y, \tau) + 2\sqrt{\tau}),$$

where $\tau = 1 - t$.

Notice, $h(y, 1) = \phi(d(y, t) - A) \cdot (0 + 0) = 0$.

We compute (notice $\tau'_t = -1$),

$$\begin{aligned}(\partial_t - \Delta)h =& \phi' \cdot (d_t - 2A)(\bar{L} + 2\sqrt{\tau}) + \phi \cdot (\bar{L}_t - \frac{1}{\sqrt{\tau}}) \\ & - (\phi'' + \phi'\Delta d)(\bar{L} + 2\sqrt{\tau}) - 2\nabla\phi \cdot \nabla\bar{L} - \phi\Delta\bar{L}.\end{aligned}$$

At minimum point of h, we have $\nabla h = 0$, i.e.

$$(\bar{L} + 2\sqrt{\tau})\nabla\phi = -\phi\nabla\bar{L};$$

then

$$\begin{aligned}(\partial_t - \Delta)h =& \phi' \cdot (d_t - 2A)(\bar{L} + 2\sqrt{\tau}) + \phi \cdot (\bar{L}_t - \frac{1}{\sqrt{\tau}}) \\ & - (\phi'' + \phi'\Delta d)(\bar{L} + 2\sqrt{\tau}) + \frac{2|\nabla\phi|^2}{\phi}(\bar{L} + 2\sqrt{\tau}) - \phi\Delta\bar{L} \\ =& (\bar{L} + 2\sqrt{\tau})(-\phi'' + (d_t - \Delta d - 2A)\phi' + 2\frac{(\phi')^2}{\phi}) + (\bar{L}_t - \Delta\bar{L} - \frac{1}{\sqrt{\tau}})\phi.\end{aligned}$$

Lemma 20.4 (I-8.3 (a))**.**

$$d'_t - \Delta d \geq -(3 - 1)\Big(\frac{2}{3}Kr_0 + \frac{1}{r_0}\Big).$$

K is the upper bound in Ricci curvature on $B(x_0, t_0, r_0)$; here we can take $K = 3 - 1$ as $r_0 = 1$; so $d'_t - \Delta d \geq -100$.

Also recall I-(7.15): $-\bar{L}_t + \Delta\bar{L} \le 2n = 6$. Combine these estimates and the property of ϕ, we see

$$(\partial_t - \Delta)h \ge -C(A)h - (6 + \frac{1}{\sqrt{\tau}})\phi.$$

So we compute (note $\tau_t' = -1$),

$$\frac{d}{dt}\Big(\log(\frac{h_{\min}(t)}{\sqrt{\tau}})\Big) = \frac{h'_{\min}}{h_{\min}} + \frac{1}{2\tau} \ge -C(A) - (6 + \frac{1}{\sqrt{\tau}})\frac{1}{\bar{L} + 2\sqrt{\tau}} + \frac{1}{2\tau}.$$

Note we proved $\bar{L} \ge -4\tau^2$, so indeed $\bar{L} + 2\sqrt{\tau} \ge 2\sqrt{\tau} - 4\tau^2$. So

$$\frac{d}{dt}\Big(\log(\frac{h_{\min}(t)}{\sqrt{\tau}})\Big) \ge -C(A) - \frac{6\sqrt{\tau}+1}{2\tau - 4\tau^2\sqrt{\tau}} + \frac{1}{2\tau} \ge -C(A) - \frac{50}{\sqrt{\tau}}.$$

Note

$$\liminf_{\tau\to 0^+}\Big(\log(\frac{h_{\min}(t)}{\sqrt{\tau}})\Big) = \log 2.$$

So

$$\Big(\log(\frac{h_{\min}(t)}{\sqrt{\tau}})\Big) \le \log 2 + \int_0^\tau C(A) + \frac{50}{\sqrt{s}}ds = \log 2 + C(A)\tau + 100\sqrt{\tau},$$

$$h_{\min}(\tau) \le 2\sqrt{\tau}e^{C(A)\tau + 100\sqrt{\tau}}.$$

Compare with (20.2), minimum of h is not reached by a barely admissible curve. So we can follow the differential inequality argument in I-8 and I-7.1. That proves (a).

21. Proposition II-6.3(b)

We argue by contradiction. The proof is divided into two steps.

Step 1 is a choosing procedure like in I-10.1 Claim 1, among all counterexamples on a manifold, we get a point $(\bar{x}, \bar{t})$ without canonical neighborhood, but with almost maximal curvature. Then in step 2 we rescale around such points and argue like in II-5.4 to get an ancient solution.

Of course, the conclusion is significant only when $r_0 \gg r(t_0)$.

Proof of (b). Assume for $K_1^\alpha \to \infty$, we have counterexamples.

Step 1.

For fixed K_1^α (so we omit α in the following), assume (x_1, t_1) is a counterexample; $t_1 = t_0$, $d(x_0, x_1, t_1) \le Ar_0$, $Q_1 = R(x_1, r_1) \ge K_1 r_0^{-2}$ while there is no canonical neighborhood at (x_1, t_1).

Assume we can find (x_2, t_2), $Q_2 = R(x_2, t_2)$, without canonical neighborhood

$$d(x_0, x_2, t_2) \le d(x_0, x_1, t_1) + \sqrt{K_1}\,Q_1^{-1/2}, \quad Q_2 \ge 4Q_1, \quad t_1 - \frac{1}{4}K_1 Q_1^{-1} \le t_2 \le t_1.$$

Once we find (x_2, t_2), we try to find (x_3, t_3) similarly. Generally, $Q_k \ge 4^{k-1}Q_1$,

$$d(x_0, x_k, t_k) \le d(x_0, x_1, t_1) + \frac{1}{1-\frac{1}{2}}\sqrt{K_1}\,Q_1^{-1/2}, \quad t_1 - \frac{1}{4}K_1 Q_1^{-1}\frac{1}{1-\frac{1}{4}} \le t_k \le t_1,$$

i.e.

$$d(x_0, x_k, t_k) \le d(x_0, x_1, t_1) + 2\sqrt{K_1}\,Q_1^{-1/2}, \quad t_1 - \frac{1}{3}K_1 Q_1^{-1} \le t_k \le t_1.$$

This procedure must stop in finite steps since $Q_k \to \infty$; once this is bigger than $r(t_0)^{-2}$ we get canonical neighborhood.

So we found $(\bar{x}, \bar{t})$, $R(\bar{x}, \bar{t}) = \bar{Q}$; so that for all (x, t) satisfying

$$d(x_0, x, t) \le d(x_0, \bar{x}, \bar{t}) + \sqrt{K_1}\,\bar{Q}^{-1/2}, \quad R(x, t) \ge 4\bar{Q}, \quad \bar{t} - \frac{1}{4}K_1\bar{Q}^{-1} \le t \le \bar{t},$$

there is a canonical neighborhood around (x,t). Remember $Q_1 \geq K_1 r_0^{-2}$, so $\frac{1}{3}K_1 Q_1^{-1} < \frac{1}{2}r_0^2$, so $\bar{t} \geq t_0 - \frac{1}{2}r_0^2$. Also,

$$d(x_0, \bar{x}, \bar{t}) \leq d(x_0, x_1, t_1) + 2\sqrt{K_1}\, Q_1^{-1/2} \leq 2Ar_0.$$

Note here it is slightly different from I-10.1 Claim 1, we might keep on finding curvature bigger and bigger, e.g go to the tip of a horn.

Step 2.

We rescale around $(\bar{x}, \bar{t})$ by $\bar{Q}$, shift time $\bar{t}$ to 0, then take limit with $\alpha \to \infty$. Since $K_1 \to \infty$, we can use the proof of Claim 2 of II-4.2 to get curvature control. So the limit $M^\infty(0)$ at $t = 0$ is a complete manifold with nonnegative curvature. As before we know $M^\infty(0)$ has bounded curvature, say C_0. We don't need the explicit value of C_0.

We remark, for sufficiently big α, every point with $R \geq 4$ within a given space distance ($\sim K_1^\alpha$), and time within $[-\epsilon\eta^{-1}C_0^{-1}, 0]$ has a canonical neighborhood. This follows from the construction of $(\bar{x}, \bar{t})$, Claim 1 of II-4.2, and the estimate in distance distort in Section 17, [11]. Now $K_1^\alpha \to \infty$, we have canonical neighborhood whenever $R \geq 4$. So we now can follow the method in II-5.4.

Since we don't have a canonical neighborhood at $(\bar{x}, \bar{t})$ we can argue like in II-5.4 to take a limit and conclude that the limit actually extend backward in time to $-\epsilon\eta^{-1}C_0^{-1}$; in particular, no surgery. We can keep on doing this and eventually get an ancient solution. That is a contradiction as in II-5.4. □

Remark 21.1. *Here, the condition $2r_0^2 < t_0$ is used to make sure the curvature pinching is good enough; we need this in order to apply Claim 2 of II-4.2. In fact, because K_1 is big, if the negative part of sectional curvature X is not small compare with $K_1 r_0^{-2}$, $\log(Xt_0)$ will be big; so we have to conclude X is small compare with $K_1 r_0^{-2}$.*

Remark 21.2. *Here, and in many other places, we argue by contradiction and end up at an ancient solution. We need to check this solution is κ-noncollapsing on* **all** *scales; compare I-11.2. In this case it is all right, because our arguments make $K_1 \to \infty$.*

22. Proposition II-6.3(c)

Proof of (c). Assume the conclusion is not true, then there are some points x with $R(x) > K_2 r_0^{-2}$.

Here $r_0 \leq \bar{r}\sqrt{t_0}$ implies that $(t_0 + 1)R(x) > K_2\bar{r}^{-2}$; so $(t_0+1)R(x)$ is a huge number. That implies $\phi < \xi$ in the pinching estimate; see our remark 7.5.

Combine with (b), we can use Claim 2 of II-4.2 (like I-12.1, one needs ϕ small in the proof). The proof is just as written in II-6.3(c). □

23. Lemma II-6.5

Exercise 12. *Where is the almost maximal volume assumption used? Is it necessary?*

A: In this lemma, I think not. We need some lower bound in volume, but not necessarily almost Euclidean. This lemma is exactly the same as Lemma I-11.6(b). As long as we have curvature bounds, we can estimate the extent to which volume is deformed, that is the content of the second part, (b).

Remark 23.1. *In I-11.6(b), the volume lower bound and the curvature estimate count on each other, for that proof, they are either both available or both unavailable.*

We remark, in 3 dimension, a lower bound in sectional curvature is the same as a lower bound in curvature operator.

Remark 23.2. *We also remark, this lemma is on smooth solutions without surgery. In fact, it is used in proving Proposition II-6.4; when it is called for, the surgeries are already ruled out. See the proof below.*

24. Lemma II-6.6

This is true even if we only have a lower bound in Ricci curvature. See Theorem 2.45 in [4].

25. Proof of Proposition II-6.4

This proposition says, if the scale r_0 is relatively small compare with $\sqrt{t_0}$, and if at t_0 a ball of radius r_0 has curvature bounded below by $-r_0^{-2}$, and almost maximal volume , then there is no surgery on this ball shortly before t_0; and we can estimate the upper bound of curvature.

We keep in mind, this will be more useful when t_0 is big.

The major motivation of this proposition is I-11.6.

Remark 25.1. *The number C_1 in the proposition should be quite big, we will indicate this in the proof below.*

Proof. **The first case**: $r_0 \le r(t_0)$. We claim $R(x, t_0) \le C_1^2 r_0^{-2}$, for any $x \in B(x_0, t_0, r_0/4)$.

If not, there is a canonical neighborhood around (x, t_0). The positive curvature neighborhoods (c and d in II-1.5) are ruled out by our design of surgeries; the remaining cases imply (x, t_0) belongs a tube or a cap.

If (x, t_0) belongs to a tube, we see around (x, t_0) the volume is too small. By $Rm \ge -r_0^{-2}$ using volume comparison we get a contradiction to the assumption that the balls with radius up to r_0 is almost Euclidean.

If (x, t_0) belongs to a cap, remember the remark after II-1.5 says that R in a canonical neighborhood are uniformly equivalent everywhere; that implies, if C_1 is suitably chosen (precisely, we just need $C_1 \gg C_2$, C_2 is the constant in the very end of II-1.5), then the cap canonical neighborhood of (x, t_0) is connected to a tube with radius at most $C^{-1/2} r_0$ with $C \gg 1$. So use Remark 4.6 in this notes, we see the volume around (x_0, t) is again way too small, and again by the volume comparison, this contradicts to the assumption that the balls with radius up to r_0 are almost Euclidean.

Exercise 13. *Is this the only place that used almost maximal volume instead of just a lower bound in volume?*

So $R(x, t_0) \le C_1^2 r_0^{-2}$. If we take $K = 2C_1^2$, $\tau = \epsilon \eta^{-1} C_1^{-2}$, by the gradient estimate $|R_t| \le \eta R^2$ in II-1.3, we see on $P(x_0, t_0, r_0/4, -\tau r_0^2)$,

$$R \le 2C_1^2 r_0^{-2} = K r_0^{-2}.$$

In particular, by assumption $r_0 > 2C_1 h$, there is no surgery in $P(x_0, t_0, r_0/4, -\tau r_0^2)$ because surgery happens in places with $R \ge h^{-2} \ge 4C_1^2 r_0^{-2}$.

The second case: $r(t_0) < r_0 \le \bar{r}\sqrt{t_0}$. We claim that we can take

$$\tau = \min\{\tau_0/2, \epsilon \eta^{-1} C_1^{-2}\}, \quad K = \max\{2K_0 \tau^{-1}, 2C_1^2\}.$$

These clearly are weaker than the values in the first case. Here K_0, τ_0 are the two constants in Lemma 6.5.

We argue by contradiction, assume the proposition is not true, that means we have a sequence of solutions, M^α, with $\bar{r}^\alpha \to 0$, $r_0^\alpha \le \bar{r}^\alpha\sqrt{t_0^\alpha}$; but the solution is not defined in $P(x_0, t_0, r_0/4, -\tau r_0^2)$, i.e. there are surgeries; or, although the solution is defined on $P(x_0, t_0, r_0/4, -\tau r_0^2)$, but the curvature estimate is violated.

Note (by assumption) $r(t_0^\alpha) < r_0^\alpha$, so among all counter examples on M^α, we can take one with (almost) minimal radius r_0^α. By the first part we can assume any smaller ball as described in the proposition satisfies the proposition. For convenience, we omit the superscript α.

Take a sub-ball $B(x_0, t_0, r_1) \subset B(x_0, t_0, r_0)$, with for instance $r_1 = r_0/2$, on which we see the conclusion of proposition holds, so the solution is defined on $P(x_0, t_0, r_1/4, -\tau r_1^2)$ and there we have $R < K r_1^{-2}$.

Because $Rm \ge -r_0^{-2}$, this is indeed a sectional curvature bound,

$$|Rm| < K r_1^{-2}.$$

Now apply Proposition 6.3 (c) to conclude that $R < K'(A\tau^{-1/2})r_1^{-2}$ for (x,t) with

$$t \in [t_0 - \tau r_1^2/2, t_0], \quad d(x_0, x, t) < A r_1.$$

Be careful, here $A\tau^{-1/2}$ plays the role of A in II-6.3(c).

Hamilton's singularity paper, [11], section 17, or I-8.3(b), tells us under Ricci flow, distance distort by at most

$$10\sqrt{K'(A\tau^{-1/2})r_1^{-2}}\Delta t,$$

on any time interval Δt in $[t_0 - \tau r_1^2/2, t_0]$. So if we take (with $r_1 = r_0/2$),

$$\Delta t = \min\{K'(A\tau^{-1/2})^{-1/2} r_1^2, \tau r_1^2/2\}, \quad A = 100 r_0/r_1 \approx 200,$$

we can guarantee that the curvature estimate

$$R < K'(A\tau^{-1/2})r_1^{-2}$$

actually holds on the parabolic neighborhood $P(x_0, t_0, r_0, -\Delta t)$. (As long as the curvature estimate works, there is no surgery).

We claim the above estimate implies $Rm > -r_0^{-2}$. If it is not true, then $X > r_0^{-2}$ (see [12] and Remark 7.5 in this notes for the notation); the pinching estimate says

$$R \ge X \log\left(e^{-3}(t+1)X\right) \ge r_0^{-2}\log\left(e^{-3}t_0 r_0^{-2}\right) > K'(A\tau^{-1/2})r_1^{-2};$$

because $r_1 = r_0/2$ and $r_0^2 t_0^{-1}$ is small enough by assumption. That is a contradiction.

Thus we can apply Lemma II-6.5 to $B(x_0, t_0, r_0)$. In particular, (b) told us the volume of $B(x_0, t_0 - \Delta t, r_0/4)$ is at least $1/10$ of the Euclidean ball with the same radius.

By Lemma II-6.6 we see there is a sub-ball B' of $B(x_0, r_0/4, t_0 - \Delta t)$ with a definite radius satisfying the assumption of this proposition (which we want to prove); by our choice of (x_0, t_0) and (the almost minimal) r_0, the proposition works for B'. Here we need almost maximal volume in an essential way, because it could be that the radius of B', or even $r_0/4$, is smaller than the critical radius $r(t_0 - \Delta t)$.

So now we repeat our argument, put B' in the position of $B(x_0, t_0, r)$, get a curvature estimate on a short time interval $\Delta t_2 \sim \tau(\operatorname{Diam} B')^{-2}$ before $t_0 - \Delta t$. More precisely, this bound is in the size of

$$K''(\operatorname{Diam} B')^{-2},$$

over $P(x_0, t_0, r_0, -\Delta t - \Delta t_2)$. This bound is worse than the previous one, but still small enough to rule out surgeries, because we have a lower bound in $\operatorname{Diam} B'/r_0$.

However, $\bar{r}^\alpha \to 0$ and the pinching estimate tell us that we actually have $-r_0^{-2}$ as a lower bound of sectional curvature, because we have a lower bound in $\operatorname{Diam} B'/r_0$.

Then apply again Lemma II-6.5 (a), and we get a much better curvature estimate; II-6.5 (b) gives a ball with radius 1/4 with big volume.

So now we returned to *exactly* the same situation before we pick the sub-ball B'. So we will do this again, and so can extend time back by the same amount like this, until we can apply Lemma 6.5 (a) and (b) no more.

As a result, we can extend time back by τr_0^2, this is a contradiction (the proposition is actually true for $B(x_0, t_0, r_0)$, so it is not a counterexample!). $\square$

26. Corollary II-6.8

This corollary says, we can drop the almost maximal volume condition in Proposition 6.4, just replace it by a *definite* lower bound in volume.

The proof just gets through as written in [16]. We note in the statement the lower bound in sectional curvature should be $-r_0^{-2}$.

27. Section II-7.1

We assume at any time, $R_{\min} < 0$, otherwise, the situation was treated in the beginning of II-6.

The main result of this section is

Theorem 27.1. *If for a fixed $r_0 < 1$, a sequence $t^\alpha \to \infty$, the parabolic neighborhoods $P(x^\alpha, t^\alpha, r_0\sqrt{t^\alpha}, -r_0^2 t^\alpha)$ converge to $P(\bar{x}, 1, r_0, -r_0^2)$ after rescale the metric by $1/t^\alpha$, then this limit solution has constant sectional curvature $-1/(4t)$ at any time $t \in (1 - r_0^2, 1]$.*

In dimension three,

$$R'_t = \Delta R + 2|\overline{\mathrm{Ric}}|^2 + \frac{2}{3}R^2 \quad \text{implies} \quad R'_{\min} \geq \frac{2}{3}R^2_{\min}.$$

So by the maximum principle, (by normalization, $R_{\min}(0) \geq -6$),

$$R_{\min} \geq \frac{3}{2} \cdot \frac{1}{\frac{3}{2}R_{\min}(0)^{-1} - t} \geq \frac{3}{2} \cdot \frac{1}{-\frac{1}{4} - t} = -\frac{3}{2} \cdot \frac{1}{t + 1/4}.$$

Because the volume V has derivative $V' = -\int R$, we see $V' \leq -\int R_{\min}$, so $V' \leq -R_{\min}V$. So

$$V' \leq \frac{3}{2} \cdot \frac{1}{t + 1/4}V, \quad \text{i.e.} \quad (V(t + 1/4)^{-3/2})' \leq 0.$$

So we define $\bar{V} = \lim_{t\to\infty} V(t + 1/4)^{-3/2}$.

Lemma 27.2. *If $\bar{V} > 0$, then*

$$\lim_{t\to\infty} \frac{R_{\min}}{-3/(2t)} = 1.$$

Proof. From the lower bound in $R_{\min}$,

$$\limsup_{t\to\infty} \frac{R_{\min}}{-3/(2t)} \leq 1;$$

so if the conclusion is incorrect, there is arbitrarily big T with $-2TR_{\min}(T)/3 < 1 - 3\epsilon$, that is

$$R_{\min}(T) \geq -\frac{3}{2} \cdot \frac{1 - 3\epsilon}{T}.$$

We observe, after T, $R_{\min}$ will not get close to $-3/(2t)$ very soon; in fact, since $R'_{\min} \geq \frac{2}{3}R^2_{\min}$, when $t > T$,

$$R_{\min}(t) \geq \frac{3}{2} \cdot \frac{1}{\frac{3}{2}R_{\min}(T)^{-1} - (t-T)} \geq -\frac{3}{2} \cdot \frac{1-3\epsilon}{T + (1-3\epsilon)(t-T)}.$$

So for all $t \in [T, \frac{4}{3}T]$, we have

$$R_{\min}(t) > -\frac{3}{2} \cdot \frac{1-1.5\epsilon}{t}.$$

So

$$V' \leq -R_{\min}V \leq \frac{3}{2}(1-\epsilon)(t+1/4)^{-1}V,$$

so on $[T, \frac{4}{3}T]$,

$$V'(t+1/4)^{-3/2} - \frac{3}{2}(t+1/4)^{-1-\frac{3}{2}}V \leq -\epsilon\frac{3}{2}(t+1/4)^{-1-\frac{3}{2}}V,$$

i.e.

$$\big(\log(V(t+1/4)^{-3/2})\big)' \leq -\epsilon\frac{3}{2}(t+1/4)^{-1},$$

so

$$\log(V(t+1/4)^{-3/2})\Big|_T^{\frac{4}{3}T} \leq -\int_T^{\frac{4}{3}T} \epsilon(t+1/4)^{-1} \approx -\epsilon\log\frac{4}{3}.$$

If $\bar{V} > 0$, $\log(V(t+1/4)^{-3/2})(\infty) > -\infty$, the above can not happen for infinitely many disjoint intervals $[T, 2T]$. That leads to a contradiction. $\square$

Let $\hat{R} = R_{\min}V^{2/3}$. We estimate

$$(\hat{R})'_t = (R_{\min})'_t V^{2/3} + \frac{2}{3}\frac{V'}{V}\hat{R} \geq \frac{2}{3}R_{\min}R_{\min}V^{2/3} - \frac{2}{3}\frac{\int R}{V}\hat{R} = \frac{2\hat{R}}{3V}\int(R_{\min} - R).$$

Since $R_{\min} < 0$ by assumption, we see the right hand side above is nonnegative. So we can define $\bar{R} = \lim_{t\to\infty}\hat{R}$. The above lemma told us, when $\bar{V} > 0$,

$$\bar{R}\bar{V}^{-\frac{2}{3}} = -\frac{3}{2}.$$

Now after the rescale (page 18 top), the parabolic neighborhood is of a definite size. Now by inequality II-(7.4), the contribution to $\hat{R}'$ from this neighborhood should go to 0 since $\hat{R}$ goes to a limit monotonically. Since the convergence is C^∞, we conclude[20], on the limit, we have $R = R_{\min}$.

In particular, if we rescale by t^{-1}, we see the limit metric is of constant scalar curvature $-3/2$. Indeed the limit is at least a parabolic neighborhood, so by the evolution equation of R, we see the sectional curvature is also constant on limit; i.e $-1/4$ at time 1 if we rescale by t^{-1}.

Note, *a priori*, we don't know $R_{\min}$ is reached in the parabolic neighborhood.

At page 18, the end of first paragraph, he said "otherwise it is vacuous". The meaning is, if $\bar{V} = 0$, after we rescale by time, the limit has zero volume. So it is not possible for rescaled parabolic neighborhood to converge.

We know surgeries does not reduce $R_{\min}$, in particular, the differential inequality for $R_{\min}$ is valid when surgeries present; also surgeries does not increase volume, so the differential inequality for volume is preserved as well. So the arguments stands with the surgeries.

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[20]Strictly speaking, here one takes derivative not with t, but with the rescaled time; and R, $R_{\min}$ are all in the rescaled (by t^{-1}) metric. The reader can easily check II-(7.4) is invariant under such a rescale.

28. Section II-7.2

Exercise 14. *In Lemma 7.2, is it a typo in (a), i.e. "wr^3" should be "$wr^3 t_0^{3/2}$"?*

Answer: apparently, yes.

Remark 28.1. *Note the notation r here has nothing to do with the canonical neighborhood parameter. We require $r < 1$, otherwise for example the conclusion (a), about the curvature, will be absurd. r is a quantity with no dimension.*

Proof of 7.2.(a). This is proved by a standard compactness argument. In order to get a convergent subsequence, we need curvature control; this is given by Corollary II-6.8.

We work on a smaller ball, make $r_0 = \min(r, \bar{r}(w))\sqrt{t_0}$, where $\bar{r}(w)$ is as in Corollary II-6.8. Observe

$$Rm > -r^{-2}t_0^{-1} > -r_0^{-2},$$

and $r_0 \gg h$ because the time is big. So the condition of Corollary II-6.8 is satisfies on $B(x_0, t_0, r_0)$. Thus we have curvature control on $P(x_0, t_0, r_0/4, -\tau r_0^2)$. By the pinching estimate that is also a sectional curvature bound. Look at the conclusion of Corollary II-6.8 we see after scale down by t^{-1} we still have a definite curvature bound.

Then the conclusion follows by using the result of 7.1.

Notice the way he wrote equation II-(7.5). *After* rescale by t, we know

$$|t^{-1}g_{ab} + 2\,\mathrm{Ric}_{ab}| \leq \xi,$$

with a, b with unit length in $t^{-1}g(t)$. So $i = t^{-1/2}a$, $j = t^{-1/2}b$ is unit in $g(t)$; thus $a = t^{1/2}i$, $b = t^{1/2}j$,

$$|\, g_{ij} + 2t\,\mathrm{Ric}_{ij}\,| \leq \xi.$$

□

Remark 28.2. *The above is proved on a tiny neighborhood of (x_0, t_0), so it is only stated at (x_0, t_0).*

Proof of 7.2.(b). We continue with the argument in (a); having a curvature control over $P(x_0, t_0, r_0/4, -\tau r_0^2)$, we apply II-6.3 to a smaller neighborhood $P(x_0, t_0, r_0'/4, -(r_0')^2)$, e.g. $(r_0')^2 = \tau r_0^2$.

We apply II-6.3 (b) first, claiming on $B(x_0, t_0, Ar\sqrt{t_0})$, any point y with $R > K_1'(A^*)r_0^2$, where

$$A^* = \frac{Ar\sqrt{t_0}}{r_0'} = \frac{Ar\sqrt{t_0}}{\sqrt{\tau}\, r_0} = \tau^{-1/2}A \cdot \max\{1, \frac{r}{\bar{r}(w)}\},$$

has a canonical neighborhood; and the canonical neighborhood must be either tube or a cap; certainly y can not be x_0 since by (a) x_0 has negative curvature while y has positive scalar curvature. We find such a point y which is closest to x_0 under the distance of time t_0; there $R = K_1'(A)r_0^2$. The pinching estimate tells us $Rm(y) \geq -r_0^{-2}$. Now because by assumption,

$$\mathrm{Vol}(B(x_0, t_0, r\sqrt{t_0})) \geq w(r\sqrt{t_0})^3, \quad r\sqrt{t_0} \geq r_0,$$

we get for $y \in B(x_0, t_0, Ar\sqrt{t_0})$,

$$\mathrm{Vol}(B(y, t_0, r\sqrt{t_0})) \geq w'(A, \frac{r}{\bar{r}(w)})(r\sqrt{t_0})^3.$$

So we can make time sufficiently big so that for the w' we can apply II-7.2 (a) that was just proved and get a contradiction; because (a) will say at y we have negative curvature while y has quite big scalar curvature by assumption.

That implies, on $B(x_0, t_0, Ar\sqrt{t_0})$, we have $R \leq K_1'(A)r_0^2$. [21]

Then we apply II-6.3 (a) to conclude that we have a uniform lower bound in volume of balls with radius r_0. The curvature bound, together with pinching, gives the lower bound in *Rm*; so we can apply II-7.2 (a) that was just proved. □

It seems we can also argue starting from II-6.3 (c), to conclude that

$$R < K'(A^*)(r_0')^{-2} = K'(A^*)\tau^{-1}(r_0)^{-2}$$

on $B(x_0, t_0, Ar\sqrt{t_0})$, where

$$A^* = \frac{Ar\sqrt{t_0}}{r_0'} = \frac{Ar\sqrt{t_0}}{\sqrt{\tau}\, r_0} = \tau^{-1/2} A \cdot \max\{1, \frac{r}{\bar{r}(w)}\}.$$

We now take time so large that the pinching estimate says

$$Rm \geq -\epsilon(r_0)^{-2} \quad \text{on} \quad B(x_0, t_0, Ar\sqrt{t_0}).$$

Now because by assumption,

$$\mathrm{Vol}(B(x_0, t_0, r\sqrt{t_0})) \geq w(r\sqrt{t_0})^3, \quad r\sqrt{t_0} \geq r_0,$$

we get for $x \in B(x_0, t_0, Ar\sqrt{t_0})$,

$$\mathrm{Vol}(B(x, t_0, r\sqrt{t_0})) \geq w'(A, \frac{r}{\bar{r}(w)})(r\sqrt{t_0})^3.$$

So we can make time sufficiently big so that for the w' we can apply II-7.2 (a) we just proved. □

Proof of 7.2.(c). We will start by showing that the conclusion is valid on $P(x_0, t_0, Ar\sqrt{t_0}, 3t_0)$, that is, extend time forward to $4t_0$. Then we repeat this argument until we reach the time $Ar^2 t_0$.

We argue by contradiction. The metric at t_0 is almost Einstein, so it will remain so for a while. Assume t_1 is the first time with

$$|2t_1 \operatorname{Ric}_{ij} + g_{ij}| = \xi$$

somewhere in $P(x_0, t_0, Ar\sqrt{t_0}, 3t_0)$. We can just assume $t_1 = 4t_0$.

First notice the curvature lower bound $Rm \geq -r^{-2}t_0^{-1}$ is satisfied (note $r < 1$ by my remark).

Since $-2\operatorname{Ric}_{ij} = t^{-1}g_{ij} + t^{-1}\xi(x,t)$, we see by integration,

$$\left| g(t_1) - \frac{t_1}{t_0} g(t_0) \right| \leq \frac{t_1}{t_0}\xi.$$

We use this to compute volume:

$$\begin{aligned} \mathrm{Vol}(B(x_0, t_1, r\sqrt{t_1})) \approx & 8\,\mathrm{Vol}(B(x_0, t_0, \frac{1}{2}r\sqrt{t_1})) = 8\,\mathrm{Vol}(B(x_0, t_0, r\sqrt{t_0})) \\ \geq & 8w(r\sqrt{t_0})^3 = w(r\sqrt{t_1})^3. \end{aligned}$$

So we can apply (b) again, use a *slightly* bigger A: notice a distance $Ar\sqrt{t_0}$ at t_0 is deformed to a distance about $Ar\sqrt{t_1}$. So we conclude at t_1,

$$|2t_1 \operatorname{Ric} + g| < \xi.$$

That's a contradiction.

We can extend time forward like this. But notice, each time the volume estimate introduced an error of size $\xi^{3/2}$; similarly the slight change in A. We can manage it as long as we have a definite number of steps to extend, e.g. in the theorem with $\Delta t = A^2 r^2 t_0$. But we can not extend infinitely many steps using the method of our proof. □

[21] We can not prove this by arguing that a tube has small volume and thus get a contradiction by volume comparison. Because when A is big, the volume comparison is not powerful enough to rule out thin tubes at a distance. On the other hand, in order to apply (a), we don't need a *explicit* lower bound in volume, as long as the time is big enough.

29. Section II-7.3

Definition. *Let $\rho(x,t)$ be the biggest $\rho > 0$ so that on $B(x,t,\rho)$,*

$$Rm \geq -\rho^{-2}. \tag{29.1}$$

First, we assume $R_{\min} < 0$, so $\rho < \infty$. We do not rule out the possibility that ρ is bigger than $\mathrm{Diam}(M)$.

Roughly speaking, within the radius $\rho(x,t)$, we expect the geometry to be relative controlled; see [4].

Lemma 29.2. *For any $w > 0$, there is $\bar{\rho}(w) > 0$ so that, if $\rho(x,t) < \bar{\rho}\sqrt{t}$, then*

$$\mathrm{Vol}(B(x,t,\rho(x,t))) < w\rho^3(x,t).$$

Proof. If not, we can apply Corollary II-6.8 to $B(x,t,\rho)$; but we first need to check the condition $\theta^{-1}h < \rho$.

If not, remember there is a point $(x^*,t^*) \in B(x,t,\rho)$ so that the minimal sectional curvature at (x^*,t^*) is $-\rho^{-2}$. Assume x^* is the point that is *closest* to x among all such points. Since θ does not depend on time (check the proof of Proposition II-6.4), because $\delta(t) \to 0$ we conclude that ρ is smaller than r, the canonical neighborhood parameter; by the pinching estimate, we conclude

$$R(x^*,t^*) \gg \rho^{-2} > r^{-2}.$$

So (x^*,t^*) has a canonical neighborhood. By the construction of surgery we see this canonical neighborhood is a tube, or a cap that is connected to a tube. So by remark 4.6 in this notes we see the volume near (x^*,t^*) is very small at any scale bigger than the radius of the tube. By the curvature lower bound (29.1) and the volume comparison [22] we get the desired volume bound.

So we apply II-6.8. Once we estimated curvature on $P(x,t,\rho/4,-\tau)$, we can use Lemma 6.3 (c) to get curvature estimate on $B(x,r,2\rho)$ by $K\rho^{-2}$. Then the pinching estimate[23] says, if time is big enough, $\bar{\rho}$ is small enough, indeed we have on $B(x,r,2\rho)$

$$Rm > -\rho^{-2}/2.$$

That contradicts to the choice of $\rho(x,t)$. □

Definition. *We define the thin part[24] $M^-(w,t)$ to be those point x with*

$$\mathrm{Vol}(B(x,t,\rho^*(x,t)) < w(\rho^*(x,t))^3;$$

here

$$\rho^*(x,t) = \min\{\rho(x,t), \sqrt{t}\}.$$

Its complement $M^+(w,t)$ is the thick part.

So if $\rho(x,t)$ is small compare with $\sqrt{t}$, x automatically belongs to the thin par M^-.

In particular, the thick part must satisfy

$$\rho(x,t) \geq \bar{\rho}(w)\sqrt{t}.$$

[22] One need to be careful here. Actually the volume comparison is not powerful enough to work for **arbitrarily small** w: see, the thin ϵ-tube within roughly has length/caliber ratio ϵ^{-1}; this is quite collapsed but perhaps not collapsed enough to imply the volume of the entire ball $B(x,t,\rho(x,t))$ is smaller than $w\rho^3(x,t)$ if w is very very small. However, we don't need w to be aribitrary small. In fact one can first prove Theorem II-7.4, a metric geometry result, i.e. manifold that is w-collapsed is a graph manifold... We just need this **critical** w. For this w, we adjust ϵ a little (**only once**!) to make the volume comparison work. This estimate is sufficient for later applications.

[23] Compare the discussion of II-6.4 in this notes

[24] Thin parts are not automatically "collapsing". There is one exceptional case we need to check in the next section.

So we can apply Lemma II-7.2 to a sequence of points in M^+ when time goes to ∞ and conclude these parts, after rescale by t, converge to a complete hyperbolic manifold. These pieces are of finite volume because $V(t)(t+1/4)^{-2/3}$ is non-increasing. Since we specify w to measure if a ball collapses, there are only finitely many hyperbolic pieces.

At this point we have a rough decomposition of the solution into a thick part (the almost hyperbolic pieces) and the remaining parts. The major topological issue is, to show the fundamental groups of the almost hyperbolic pieces injects into the fundamental group of the whole solution at any time.

This is proved in [12]. In [12], Hamilton classified nonsingular solutions to the normalized Ricci flow. The adjustment we need here, if any, is minor. What is needed is, the persistence of hyperbolic pieces, and that the boundary tori are incompressible.

To prove the persistence of hyperbolic pieces, Hamilton used hyperbolic rigidity and harmonic map on manifolds with boundary; none of these are directly related to Ricci flow. Specifically, in order that argument gets through, one just need the following fact: assume we have a family of metrics parameterized by time t; if a sequence of points, (p_i, t_i), each has a neighborhood that is ξ-close to a hyperbolic ball, then for any subsequence of (p_i, t_i) which we denote also by (p_i, t_i), there exist neighborhoods of (p_i, t_i) that converges to a *complete* hyperbolic manifold as $i \to \infty$. In our situation, we consider the manifolds with metric $t^{-1}g(t)$; the above mentioned requirement follows from II-7.1 and II-7.2.

Next consider the incompressible issue. We are now dealing with a family of metric $h(t) = t^{-1}g(t)$. We see

$$h' = -t^{-1}h - 2t^{-1}\,\mathrm{Ric}\,.$$

For convenience, all the notions (curvatures, etc) in this argument are respect to h. Follow the argument in [12]. Denote by A the area of the (supposed) minimal disc D whose boundary ∂D lies in a torus T, so that ∂D is a primitive element of $\pi_1(T)$ that generates (the supposed) kernel of $\pi_1(T) \to \pi_1(M^c)$; M^c is the complement of the hyperbolic piece to which T belongs. T is of *constant mean curvature*, and almost constant second fundamental form, and of *definite area*. We need these to make A a well-defined quantity.

Notice, this torus T may move with respect to time, in order to preserve constant mean curvature and constant area. The derivative of metric here is $-t^{-1}(h + 2\,\mathrm{Ric})$; by II-7.2 we know the tensor $h + 2\,\mathrm{Ric}$ is already very small. Also we know the persistence of hyperbolic pieces. All these tells us the V term in [12], section 11, is of size $o(t^{-1})$. (Recall, the proof of existence of such a torus T uses the implicit function theorem, as described in the footnotes to [12] in [1]. Thanks Prof. Chow for explaining these issues).

We compute the variation of area as in [12]. Denote by D the minimal disc,

$$A' \le -\int_D \frac{1}{t}dA - \int_D \frac{R^h}{2t}dA - \frac{2\pi}{t} + \frac{1}{t}\int_{\partial D} k_g + o\Big(\frac{1}{t}\Big).$$

Here k_g is the geodesic curvature of ∂D. It almost equals to $1/4$ since the torus is of almost constant second fundamental form (check how to rescale!). The R^h above is the scalar curvature of the metric $h = t^{-1}g$; by II-7.1 it almost equals to $-3/2$. So in conclusion we get

$$A'_t \le \frac{1+2\epsilon}{4t}L - \frac{1-\epsilon'}{4t}A - \frac{2\pi}{t} + o\Big(\frac{1}{t}\Big).$$

L is the length of ∂D. So finally we follow the argument in [12] section 12 to bound length L by area A, we conclude

$$A'_t \le -\frac{2\pi - \xi}{t}.$$

This is not possible because the right hand side is not integrable.

These arguments works with the surgeries. In fact, the surgeries are essentially breaking off necks. They happen when a long, thin tube pinches off in the center or (degenerate case) one end. Assume a minimal disc passes through a thin tube because. Look at *all* the intersections of the minimal disc with a center S^2; these are all genericly circles (if not, take some generic S^2 close to the center one...). Consider *all* out most circles in the disc; The "outside" of these circles is a disc with many holes; we observe that this set remain connected after a surgery because we have counted all intersections, on both sides of surgery. So now we can just fill in these circles locally near the center S^2, as we have seen these remain connected to the T^2 we are studying even after a surgery. So we see doing surgery actually reduces the area of A. Compare [17], we remark after sufficiently long time, all surgeries are topologically trivial, thanks the prime decomposition.

In fact I don't think that a minimal disc has to pass through a neck.

30. Section II-7.4: the application

Remark 30.1. *We will apply theorem II-7.4 to the thin part of a solution to Ricci flow,* **after** *we rescaled the solution down by t^{-1}. (So in (2) the curvature is about $1/4$).*

The sectional curvature lower bound in II-7.4(3) should be $-r_0^{-2}$.

We need to explain that the thin part satisfies the assumption of theorem II-7.4, with only one possible exceptional case that is easy to treat.

Having the persistence of hyperbolic pieces, we truncate these pieces at boundary tori with sufficiently small area (so that the remaining parts are thin in the sense of II-7.3), and with almost constant second fundamental form; this is possible by the persistence of hyperbolic pieces. In particular, the remaining (thin) parts has convex boundary.[25]

The thin part satisfies (1) by definition, or almost, except the part concerning the diameter of the manifold; see the following paragraphs for this issue. And (2) is also satisfied since the boundary between thin and thick part (on a *cusp*) can be chosen like this, the metric is almost hyperbolic. (3) apparently follows from Corollary II-6.8 and Shi's estimate [18], [19].

Now let us look at (1) more carefully. **Note here the metric is** $t^{-1}g(t)$.

If (1) is not satisfied, we claim M has no hyperbolic pieces, that is, there is no thick part, and M is a compact manifold with no boundary so that

$$w\rho^3 \geq \mathrm{Vol} \geq c\,\mathrm{Diam}^3, \quad Rm \geq -\rho^{-2} \geq -\,\mathrm{Diam}^{-2}\,.$$

In fact, we have the almost cusps between a thin part and a thick part, so when there is a thick part which is not the whole manifold, there is a long cusp; one can arrange that the diameter of the "thin" part is more than 100, while the curvature near cusp is $-1/4$. That contradicts to $Rm \geq -\,\mathrm{Diam}^{-2}$.

Moreover, c stays away from 0. Otherwise we can apply Theorem II-7.4 to show that M is a graph manifold.

We also remark that Diam is smaller than $\bar{r}$ (use metric $t^{-1}g(t)$); in fact, it should go to 0 at $t \to \infty$. Otherwise, M would converge to a hyperbolic manifold.

Lemma 30.2. *For any K_0, c, there is a τ so that, if a manifold N satisfies*

$$\mathrm{Diam}(N) = 1, \quad \mathrm{Vol}(N) \geq c, \quad -\tau \leq Rm \leq K_0,$$

and also Perelman's condition II-7.4(3), then N admits a metric with nonnegative sectional curvature.

[25] I don't know why this is needed in the proof of theorem II-7.4.

Proof. Compactness. Probably one does not need II-7.4(3). □

The K_0 above is the one in II-7.4(3).

Now if $\rho\sqrt{\tau} >$ Diam, If we rescale the "exceptional" manifold by Diam^{-2}, we can apply the Lemma so the topology of M is fully understood.

Otherwise, by taking a even smaller w, we see $\mathrm{Vol}\,\mathrm{Diam}^{-3}$ is small. For more about this, see the next section. So again we can apply Theorem II-7.4 to M.

31. Assemble the proof: A brief remark on the constants

Let us try to assemble the proof.

One first goes through the first paper.

Then prove Theorem II-7.4 and get a critical $\omega_0 = w^\alpha$; that is, if for all $p \in M$, there is ρ so that

$$Rm \geq -\rho^{-2} \text{ on } B_\rho(p), \quad \mathrm{Vol}(B_\rho(p)) \leq \omega_0 \rho^3,$$

and assume II-7.4 conditions (2), (3), then M is a graph manifold. The proof is purely metric geometry.

For this ω_0, we go through the arguments II-1 to II-6, and find K_0 so that if

$$Rm \geq -\rho^{-2} \text{ on } B_{2\rho}(p), \quad \mathrm{Vol}(B_\rho(p)) \geq \omega\rho^3, \quad \rho \ll \sqrt{t},$$

then for t big enough, $Rm < K_0$ on $B_\rho(p)$.

Then put $c = \omega_0$, K_0 in Lemma 30.2 and find the corresponding τ.

Now define the thin part as: (under metric $t^{-1}g(t)$) for the biggest ρ so that

$$Rm \geq -\rho^{-2} \text{ on } B_{2\rho}(p),$$

we have

$$\mathrm{Vol}(B_\rho(p) \leq \tau^{\frac{3}{2}}\omega_0\rho^3.$$

This, first of all settles the "exceptional case". In fact, if we cannot apply II-7.4, it must be

$$\tau^{\frac{3}{2}}\omega_0\rho^3 \geq \mathrm{Vol} \geq \omega_0\,\mathrm{Diam}^3,$$

then we can apply Lemma 30.2. Note in II-7.4 (1) and (3) Perelman used different symbols (w^α vs w').

In the arguments of II-1 through II-7, one need to be careful about which constant depends on which.

After the proof in II-4, II-5, we are free to further reduce the value of $\delta(t)$ when necessary.

Many other constants, e.g. the K in II-6.4, κ in II-6.3, are insensitive to time t. And they will not get worse if we reduce δ.

For instance, in the beginning of the proof in II-6.4, one need to reduce δ further so that $\delta \ll K^{-1}$ in order that a curvature as big as Kr^{-2} is still not big enough to call for a surgery. (In this case K comes from earlier works, e.g. II-6.5 and volume comparison).

It is possible one can make sense of "weak solution" of the Ricci flow and take $\delta = 0$.

Appendix: Theorem II-7.4 and the method of Shioya-Yamaguchi

In April 2003, Shioya and Yamaguchi posted a paper, [22], with a proof of Perelman's theorem II-7.4. In fact, the major ideas and the (most important) local case are already available in their earlier works, [7], [25], [21]. Here we give a sketch of their method.

A collapsing 3-manifold looks like a 1 or 2 dimensional object. So first we need to study Alexandrov spaces of dimension 1 (that's trivial) and 2, as the limits of collapsing sequences. It turns out there are some fiber structure on at least parts of the manifolds that converge to the "nice" parts of the limit. One then try to glue these local structures to get global information.

Sometimes we will not distinguish notations on manifolds and on the limit Alexandrov spaces; this should not cause much confusion.

1. The critical points in the sense of Grove-Shiohama.

Consider the distance function $d(p, \cdot)$ over a manifold. If for all vector T at x, there is a minimal geodesics γ from x to p so that

$$\angle(\gamma'(0), T) \leq \frac{\pi}{2},$$

we say x is a **critical point** (for $d(p, \cdot)$).

Proposition 31.1. *If there is no critical points for $d(p, \cdot)$ over $B_r(q)$, then there is a set $E \supset B_r(q)$ that is diffeomorphic to $S \times \mathbf{R}$, where S is some $n-1$ surface.*

In fact, without critical points, one can construct a vector filed V that is "almost in the gradient direction of $d(p, \cdot)$". The diffeomorphism is obtained by following the integral curves of V.

Proposition 31.2. *There is no critical points for $d(p, \cdot)$ near the middle of a minimal geodesic connecting p and q.*

Otherwise, we get two minimal geodesics from x to p and to q, with small angle at x, that contradicts to the Toponogov theorem.

2. Alexandrov spaces. [3].

Assume X is an Alexandrov space with curvature at least K, for example, any limit of Riemannian manifolds with sectional curvature at least K. For convenience, we assume $K = 0$.

Assume γ_1, γ_2 are two minimal geodesics, $\gamma_1(0) = \gamma_2(0) = x$. Define[26]

$$\alpha(t_1, t_2) = \arccos\Big(\frac{t_1^2 + t_2^2 - d(\gamma_1(t_1), \gamma_2(t_2))^2}{2t_1t_2}\Big).$$

The Toponogov theorem implies α is a nonincreasing function of t_1, t_2. So it makes sense to define the **angle** between γ_1, γ_2 at x as the limit of $\alpha(t_1, t_2)$ when $t_1, t_2 \to 0$.

A tangent cone at $x \in X$ is defined as the pointed Gromov-Hausdorff limit of $(X, x, r_i^{-2} d)$, where d is the distance on X, $r_i \to 0$. This limit does not depend on the choice of the sequence r_i (because we can define the angle between γ_1, γ_2...).

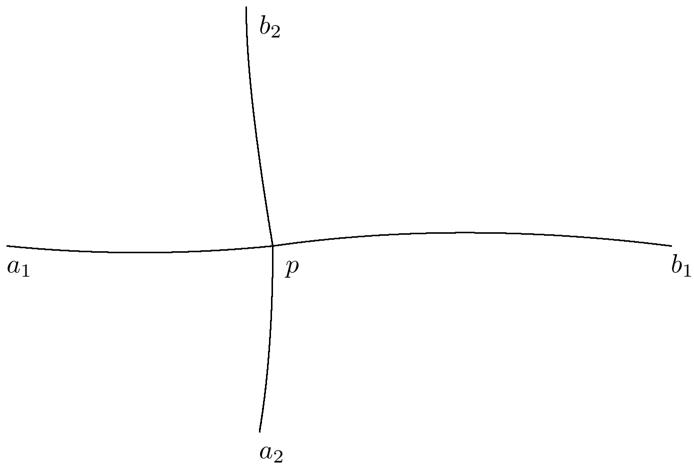
A point $x \in X$ is **regular**, if the tangent cone at x is $\mathbf{R}^k$.

Then define a **strainer** at p: it is a set of points $a_1, a_2, ..., a_n; b_1, b_2, ...b_n$, so that all the distances $d(p, a_i), d(p, b_i)$ are comparable, and for $i \neq j$,

$$\frac{d(a_i, a_j)}{\sqrt{d(a_i, p)^2 + d(a_j, p)^2}} \approx 1, \quad \frac{d(b_i, b_j)}{\sqrt{d(b_i, p)^2 + d(b_j, p)^2}} \approx 1, \quad \frac{d(a_i, b_i)}{d(a_i, p) + d(b_i, p)} \approx 1.$$

[26] If $K < 0$, we have to use the law of cosine on space with constant curvature K.

The biggest number r so that there is a stainer at p with $r = \min\{d(a_i, p), d(b_j, p)\}$ is called the **strainer radius** at p.



Proposition 31.3. *If there is a strainer at p of radius r, and the curvature is at least $-\epsilon(n)^2 r^2$, then there is a bi-Lipschitz map from a set containing $B_{\epsilon(n)}(p)$ in p to a ball in R^n.*

In fact, the map $x \mapsto (d(a_1, x), d(a_2, x), ..., d(a_n, x))$ gives the bi-Lipschitz map. You can prove the case $n = 2$ as an exercise (use the Toponogov theorem...).

3. The geometry of 2-Alexandrov spaces. [3], [20], [21].

Proposition 31.4. *A 2-Alexandrov space is a topological manifold of dimension 2, possibly with boundary.*

For all $\delta > 0$, let $S_\delta \subset X$ be the set of points x in X so that the tangent cone at x is a cone over a circle with perimeter $l \leq 2\pi - \delta$. [27]

Proposition 31.5. *There are only finitely many S_δ within finite distance.*

One can prove this directly. However, there is a version of 2-dimensional Gauss-Bonnet theorem for Alexander spaces; [22], a singular point carries a "curvature mass" of $2\pi - l$. You can think the case that X is the limit of a sequence of surfaces. So indeed we can estimate precisely how many points belong to S_δ.

We define the **essential boundary** $\partial_* X$ to be those $x \in X$ so that the tangent cone at X is a cone over a closed segment (of length at most π). Sometimes we need to cut X by some distance functions, the boundary pieces formed this way are not regarded as essential boundary.

4. The soul theorem in dimension 3. [28] Assume M is a three dimensional complete manifold of nonnegative sectional curvature. Then we have the soul theorem of Cheeger-Gromoll.

If the soul is two dimension, it must be S^2, RP^2, T^2 or the Klein bottle. If the soul is S^2, M must be a metric product $S^2 \times \mathbf{R}$ by the splitting theorem; if the soul is RP^2, M must be $S^2 \times \mathbf{R}/Z^2$, by going into the double covering. If the soul is T^2 or Klein bottle, M must be flat. For example, there is only one R bundle on T^2 so that the total space is orientable; so M has two ends and thus splits.

[27] If $l = 2\pi$, then x is a regular point.

[28] We state it for the smooth case. Actually, this is true for 3-Alexandrov spaces with nonnegative curvature; [21]. however, with the help of condition (3) in Perelman's theorem II-7.4, we can avoid the general case.

Assume the soul is S^1, then the manifold must be diffeomorphic to $S^1 \times E$; E is a complete surface that is diffeomorphic to R^2. This is because the lift of S^1 is a line (notice a soul is totally convex); so there is a splitting in the S^1 direction. However the metric need not to be a global metric product.

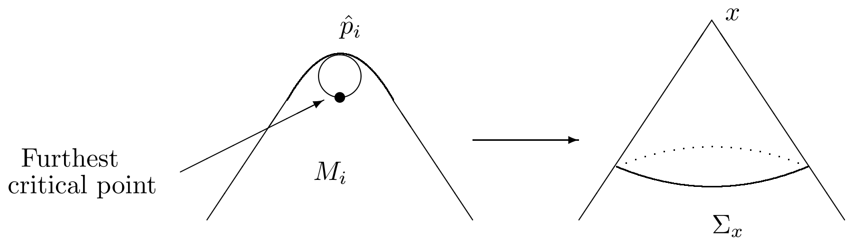
Assume the soul is a point, p, then M is diffeomorphic to R^3.

5. The Shioya-Yamaguchi limit. [21].

Assume a sequence of manifolds converges to an Alexandrov space of lower dimension (i.e. the collapsing case). We first focus on singular points x of X, i.e. the tangent cone at x is not R^k.

We assume $B_R(x)$ is very close to (after rescale by R^{-1}) the corresponding ball B_R in the tangent cone at x.

The standard method of analyzing a singularity is to find a rescaled limit. The first technical issue is, to choose suitable base point $\hat{p}_i \in M_i$, with $\hat{p}_i \to x$ in the Gromov-Hausdorff sense. We want $\hat{p}_i$ on the "peak" as shown in the picture; this is achieved by maximizing the average distance to Σ_x, the cross section of the tangent cone; for details see [21].



Let δ_i be the distance from $\hat{p}_i$ to the *furthest* critical point of $d_i(\hat{p}_i, \cdot)$ *within distance* R. We know $\delta_i \to 0$, because $(M_i, \hat{p}_i)$, at scale up to R, is very close to a metric cone. So critical points at distance comparable to R are ruled out by the Toponogov theorem, c.f. Proposition 31.2.

Then take limit of $(M_i, \hat{p}_i, \delta_i^{-2} dx_i)$. Such a limit, which is an Alexandrov space with nonnegative curvature, is called a **Shioya-Yamaguchi limit** (SY limit) associated with the singularity x. In particular, the ball $B_2(\hat{p}_i)$, contains all the information about the topology on M_i that is near the singular point x in the Gromov-Hausdorff sense.

One can check the dimension of an SY limit is at least one more than the dimension of X. Roughly speaking, our choice of $\hat{p}_i$ forces the furthest critical point lies in a direction almost perpendicular to the direction that leads to Σ_x, that gives us an extra dimension. In our situation M_i are of dimension 3, if X is of dimension 2, then a SY limit must be of dimension 3; if X is of dimension 1, then a SY limit has dimension at least 2.

6. The fibration theorem. [25].

Proposition 31.6 (Yamaguchi)**.** *Assume* $M_i \xrightarrow{d_{GH}} X$*, with curvature bounded from below by* -1*. Assume the strainer radius on* X *is at least* δ*; then for sufficiently big* i *there is a fibration on* M_i*.*

Consider the Hilbert space $L^2(X)$. For each $p \in X$, we get a function $f_X(p) \in L^2(X)$ that is defined as

$$f_X(p)(x) = h(d_X(x,p)),$$

where $h \geq 0$ is a function supported near 0; the size of this support is smaller than the stainer radius. So $f_X : X \to L^2(X)$ is certainly a bi-Lipschitz embedding. One can check, even though it is not a smooth embedding, one can define "rough" tangent bundle and a "rough" normal bundle of $f_X(X) \subset L^2(X)$; in particular, for points sufficiently close to $f_X(X)$ we can define a "projection" π onto X.

Similarly, for each $m \in M$, we get a function $f_M(m) \in L^2(X)$ that is defined as

$$f_M(m)(x) = h(d_M(m, \phi(x))),$$

ϕ is a Gromov-Hausdorff approximation from X to M. f_M is generally not an embedding of M into $L^2(X)$; however we still can consider the map $f_X^{-1} \circ \pi \circ f_M$ from M to X; one can prove that is a submersion and gives the fibration, [25].

In the case X is of dimension 2, the fiber is clearly S^1; when X is of dimension 1, the fiber is either S^2 or T^2, this can be seen directly by taking limit similar to the SY limit and reduce the study to 2-dimensional limit spaces; one can also check the first Betti number of fiber is at most 2, as in [24], [21], [25].

7. Local structure over 2-Alexandrov spaces.

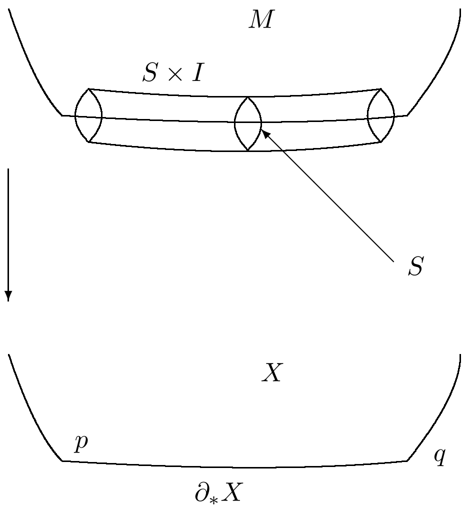
Close to any regular points in X, there is a local S^1 fibration on M. Near a singular point x, we can take a small r so that $B_r(x)$, in its own scale, is close to the tangent cone at x. Cones of dimension 2 is regular away from the pole, so there is a fibration over $\partial B_r(x)$, thus we get a $\mathbf{T} \approx T^2$ over $\partial B_r(x)$.

We then study the SY limit Y at x, it is a complete 3-manifold with nonnegative curvature, by Perelman's condition II-7.4(3). Notice the boundary of big balls in Y is a torus, T^2, the soul theorem implies Y must be $S^1 \times E$ where E is a surface diffeomorphic to $\mathbf{R}^2$. So $\mathbf{T}$ actually bounds a solid torus so we can get some [29] Seifert fiber structure over $B_r(x)$.

So we have a Seifert fibered structure over the interior of X.

Next study the part close to $\partial_* X$. We first divide $\partial_* X$ into many small pieces, so that each piece is almost "straight". Let $\overline{pq}$ be such a piece. We also make the scale of $\overline{pq}$ so small that there is no point of S_δ (see the definition before Proposition 31.5) near $\overline{pq}$.

So the function $d(p, \cdot)$ does not have critical points on $\overline{pq}$, we can get an open set in M that is close to $\overline{pq}$, and diffeomorphic to $S \times I$, where S can be taken as a piece of a level surface of $(p, \cdot)$ near the middle of $\overline{pq}$.

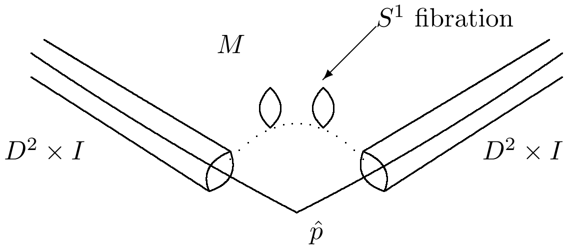


Because there is no point of S_δ near $\overline{pq}$, the fibration theorem, which is valid on regions that is not too close to $\overline{pq}$, applies. In particular, there is a deformation retraction map from $B(S, r)$ to S, here r is comparable with half of the distance form p to q, but much bigger than the size of S.

[29] If this singularity is not too singular, i.e. the angle is more than π, we know this fill in fibration is trivial; and it agrees with the boundary fibration.

Then the Margulis lemma, [21] implies $\pi_1(S)$ is almost nilpotent. There is only one surface with nonempty boundary has an almost nilpotent fundamental group, that is D^2. So S is a disc D^2.

Near a corner point p, we study the SY limit at p. The set $\partial B_\mu(\hat{p})$, will cut out two discs near $\partial_* X$, while the boundary of these discs will be connected by an annulus in $\partial B_\mu(\hat{p})$; this annulus is formed by the S^1 fibration on regions away from $\partial_* X$. So $\partial B_\mu(\hat{p})$ is a sphere, see the picture:



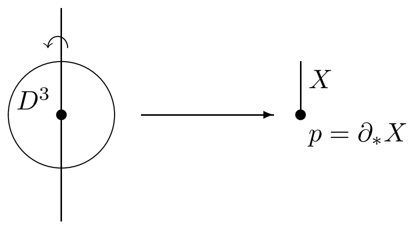
If the boundary of big balls in the SY limit is a sphere, the soul theorem implies the limit is diffeomorphic to $\mathbf{R}^3$. Thus $\partial B_\mu(\hat{p})$ bounds a solid ball.

So the structure near $\partial_* X$ is, there is an open set that is diffeomorphic to $\partial_* X \times D^2$, and is Gromov-Hausdorff close to $\partial_* X$, so that for each $x \in \partial_* X$, $x \times \partial D^2$ is an S^1 fiber on the regular part of X. Roughly speaking, the S^1 fibers near $\partial_* X$ are "filled in" by discs.

8. Local structure over 1-Alexandrov spaces.

Assume the limit space X is of dimension 1. Then on the regular part there is an S^2 or a T^2 fibration. If p is a singular point, then X is a segment or a ray starts at p. Take an SY limit Y at p. If Y is of dim 3, then by the soul theorem it is diffeomorphic to $\mathbf{R}^3$, or $S^2 \times \mathbf{R}/Z_2$; $S^1 \times \mathbf{R}^2$, or $K^2 \tilde{\times} \mathbf{R}$; depends on whether the fiber on the regular part is S^2 or T^2. So the structure near p is just gluing D^3, or $S^2 \times I/Z_2$; $S^1 \times D^2$, or $K^2 \tilde{\times} I$ to the regular ends.

When the fiber on the regular part is T^2, then gluing Y to the regular T^2 fibration gives a graph manifold. If the fiber is S^2, we remark there is one S^1 action. For example, if the SY limit is $\mathbf{R}^3$, near the singularity p, it is like the following:



The center of the ball D^3 projects to p; the fibers are the concentric spheres in D^3. We remark the fixed points in the action is a curve (the diameter of D^3). There is a similar picture if the SY limit is $S^2 \times R/Z_2$. This picture describes the action only, in reality the size of the S^2 fiber is small.

Locally this does not give a Seifert fiber structure. However, when we do a global gluing, there will be a graph manifold structure. See the subsequent subsection "Finding global structures".

If the SY limit Y is of dim 2, by the argument in the previous subsection, basically over p, we see some Seifert fiber structure over Y (with or without boundary); and it is connected to a piece with S^2 or T^2 fibers over a segment. The structure that connects these two pieces will be discussed in a subsequent subsection "Finding global structures". [30]

9. The building block.

Proposition 31.7. *There is a $w > 0$ so that, if $Rm \geq -1$ on $B_2(p)$,*

$$\mathrm{Vol}(B_2(p)) < w,$$

Then there is an Alexandrov space (X_p, x), with or without boundary, so that for some $1 \leq r \leq 2$,

$$d_{GH}(B_r(p), B_r(x)) < \epsilon;$$

and on $B_r(p)$ there is a structure as described in the last two subsections; moreover, there is no S_δ points in the σ neighborhood of $\partial B_r(x) \subset X$.

Remember, S_δ are points at which the tangent cone is a cone over a circle with perimeter less than $2\pi - \delta$.

The proof is, of course, by taking limit.

This result is suitable for our purpose, because in II-7.4 there is only a local lower bound in curvature, we need to rescale. In particular there is no bound for diameter, it can well happen that one region looks like 2 dimensional while another region that is far away looks like 1 dimensional.

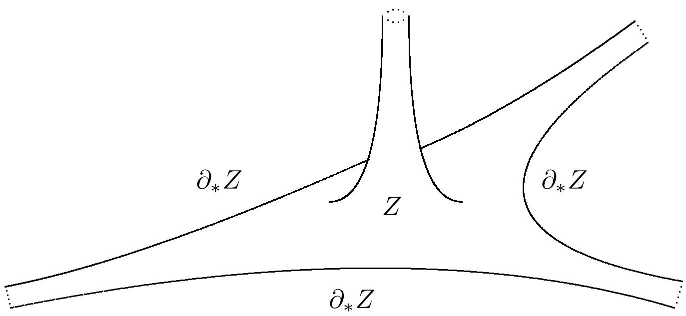
So in order to glue the above building blocks together, we need to glue structures over 2 dimensional Alexandrov spaces first, then glue structures over 1 dimensional Alexandrov spaces, finally we glue over the "transition regions" between 2 dimensional Alexandrov spaces and 1 dimensional Alexandrov spaces.

10. Gluing structures with the same dimension.

Assume p and q are so that the dimensions of X_p and X_q (as in the previous subsection) are the same. The main idea is, on the overlap $B_r(p) \cap B_{r'}(q)$, the fibers roughly over $\partial B_r(x)$ are isotopic; this is because, by construction, points are almost regular near $\partial B_r(x)$; so one can use the gradient of distance functions to flow one fiber to another.

So by perturbing the fibers near $\partial B_r(x) \subset X_p$, one can glue the fibers together. One can keep on doing this. More precisely, let U_2 be a connect component of those p so that the space X_p in Proposition 31.7 is of dimension 2; then U_2 has the following structure:

There is a (compact, with boundary) 2-surface Z, whose boundary ∂Z is divided into two parts, one part is $\partial_* Z$. There is a Seifert fibered space $\hat{U}_2$ over Z (viewed as a closed set), and U_2 is obtained by fill in the S^1 fibers over $\partial_* Z$ with discs D^2.



[30] With more effort, e.g. a more careful analysis of the structure over singularities and a finer classification of 2-Alexandrov spaces with nonnegative curvature, [21], one actually sees in this case the structure is a solid D^3 or $RP^3 - D^3$ glues to a $S^2 \times I$, or $S^1 \times D^2$ or $K \tilde{\times} I$ glue to $T^2 \times I$.

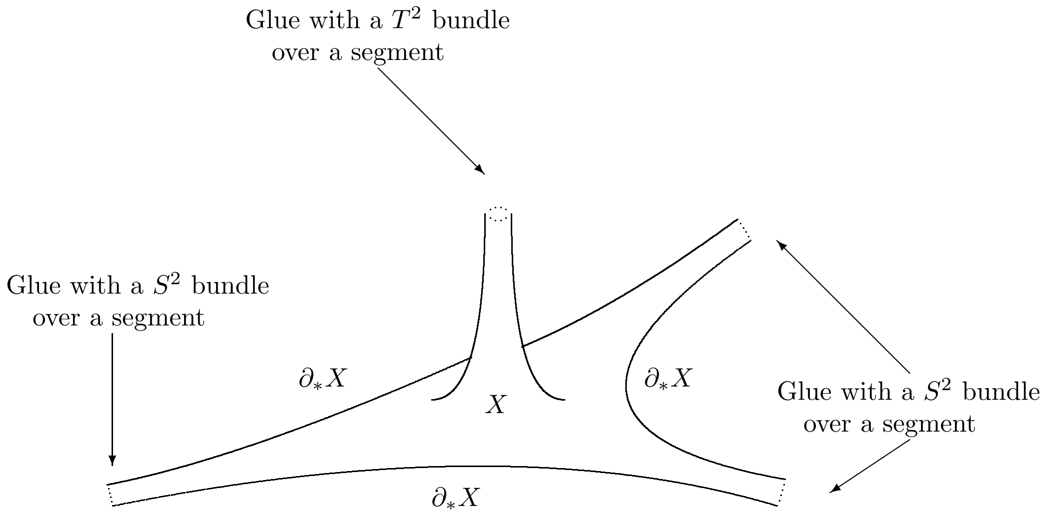
So if there is a piece of $\partial Z - \partial_* Z$ that is a circle, there is a corresponding T^2 boundary for U_2; besides that, each component of $\partial Z - \partial_* Z$ is a segment; and there is a corresponding S^2 boundary for U_2.

Similarly, let U_1 be a connect component of those p so that the space X_p in Proposition 31.7 is of dimension 1; then U_1 has the following structure:

There is a piece E that is diffeomorphic to $S^2 \times I$, or $T^2 \times I$. If E is not all of U_1, then there is one, or two spaces, each of these are either one of D^3, $S^2 \times I/Z_2$, $S^1 \times D^2$, $K^2 \tilde{\times} I$, or with the structure as U_2 described above, each space has only one boundary component, which is glued to end (or ends) of E. See the next subsection.

11. Finding global structures.

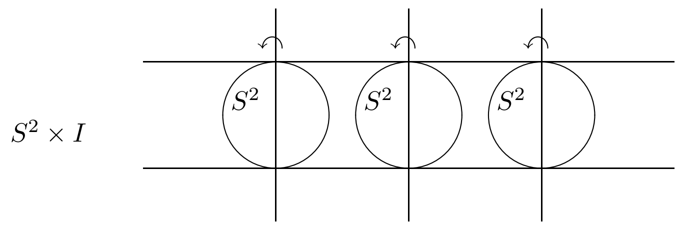
The gluing between U_1 and U_2 is clear, one just match the S^2 or T^2 boundary. The gluing of T^2 is of course described by elements of $SL_2(\mathbf{Z})$; but we don't need to worry about this, thanks the very definition of graph manifolds.



So the question is, how to find the graph structure when we glue U_2 with the S^2 bundles over I.

Assume one S^2 boundary piece of U_2 is glues to a $S^2 \times I$. Then the other end of this $S^2 \times I$ is glued with D^3, or $RP^3 - D^3 = RP^2 \tilde{\times} I$, or another S^2 boundary piece of (probably another) U_2.

Now we construct an S^1 action on the union of all these $S^2 \times I$, D^3, $RP^3 - D^3 = RP^2 \tilde{\times} I$, and the region near $\partial_* X$. Remember, the structure near $\partial_* X$ is $\partial_* X \times D^2$, and each $x \times \partial D^2$ is a fiber. So we can put an S^1 action on $\partial_* X \times D^2$, by rotating the D^2. Note this action has fixed points, which is diffeomorphic to (and Gromov-Hausdorff close to) $\partial_* X$. We can arrange so that this action extends to the boundary S^2 pieces of U_2. (We only consider orientable manifolds!) Then extend it to the $S^2 \times I$:



Note the fixed points (the north poles and the south poles) form two intervals. We can extend this to the other U_2 piece, or to D^3 or $RP^3 - D^3 = RP^2\tilde{\times}I$ that is glued to the other end of $S^2 \times I$.

The fixed points of this action is one (or more) circle(s), in naked eyes these circles "link the $\partial_* X$ pieces".

Remove a (very thin) solid torus around the fixed point sets. The remaining part is some Seifert fibered spaces glued along the T^2 boundaries with a set that admits an S^1 action which is almost free, except there are possible fixed points for $e^{i\pi} \in S^1$: these arise from the action on $RP^3 - D^3 = RP^2\tilde{\times}I$. These fixed points form some disjoint circles, together with the S^1 orbits one gets a Seifert structure.

So the manifold admits a graph structure. Here the T^2 *between* Seifert spaces might not be incompressible, e.g. there might be solid tori; but this does not effect the correctness of geometrization. [31]

Appendix: DeTurck's trick

We review DeTurck's trick.

Assume $V(x,t) = V^i \partial/\partial x_i$ is a vector field. It generates a one-parameter diffeomorphism group $\phi(t)$. Assume g is a metric depends on time t, one computes

$$\frac{d(\phi_t^* g)}{dt} = \phi_t^* \Big(\frac{dg}{dt}\Big) + 2\phi_t^*(\delta^* v),$$

here v is the dual 1-form of V, $\delta^* v$ is a symmetric 2-tensor with [32]

$$\delta^* v(\mathbf{x}, \mathbf{y}) = \frac{1}{2}(\nabla_{\mathbf{x}} v(\mathbf{y}) + \nabla_{\mathbf{y}} v(\mathbf{x})).$$

This can be used to change the Ricci flow equation into a strict parabolic equation. Recall, for a variation of metric, $g + sh$,

$$\begin{aligned}(2\ \mathrm{Ric})'_s(X,Y) = &- 2\delta^*\delta h(X,Y) - \mathrm{Hess}_{\mathrm{Tr}\ h}(X,Y) + \nabla^*\nabla h(X,Y) \\ &+ \mathrm{Ric}(X,e_i)h(e_i,Y) + \mathrm{Ric}(Y,e_i)h(e_i,X) - 2h(R(e_i,Y)X,e_i).\end{aligned}$$

Observe, $\delta^*\delta f g = -\delta^* df = -\mathrm{Hess}_f$, we see

$$(2\ \mathrm{Ric})'_s(X,Y) = - 2\delta^*\delta(h - \frac{1}{2}\mathrm{Tr}\ h)(X,Y) + \nabla^*\nabla h(X,Y)...$$

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[31] The graph manifold here is defines by gluing Seifert fibered spaces together. In the gluing, if boundary torus T are not imcompressible, we can do the following:

Assume T is a seperating torus. Find a disc D with ∂D embedded in T. Find an embedded I-bundle E on D so that $E \cap T$ is exactly the I-fibers over ∂D.

Cut the manifold along $T \cup \partial E - (T \cap \partial E)$. We see this is just cut along a S^2. The first case is, this S^2 does not bound a ball on either side. Then clearly such a splitting cut M into a nontrivial connect sum, $M = M_1 \# M_2$. Both of them are graph manifolds. In fact, M_2, the one comes from the "D side of T", is obtained by cutting a unknotted solid torus from S^3 and glue in the originally pieces in M that lies on D side of T; M_1, the other piece is obtained by glue in to M a solid torus to replace the piece of M that is on "D side of T".

If this torus is not seperating, then we can still do the above, cut along the surface above and get a connected manifold with two S^2 boundary pieces; we can glue in balls to these S^2 pieces, this time the result is just to remove a $S^2 \times S^1$ from M.

This cannot continue for ever, by the prime decomposition.

The remaining case is, T bounds a solid torus, we can assume the fiber structure does not extend. That means, the fiber structure is the meridian circles of T^2. Consider the part of M bounded by T that is not a solid torus; that part has a base surface, which is just the S^1 action quotient.

Draw a curve on that base surface with end points touching T, cut, we will get another connect sum, or the whole thing is a lens space.

[32] See, δ^* is the adjoint of the divergence operator $\delta w = -\nabla_{e_i} w(e_i, \cdot)$. Another interpretation is the Lie derivative $\delta^* v = L_V g/2$.

It is because of the first term on r.h.s. that the equation is not strict parabolic; see [8]. DeTurck's trick is, to find some form v depends on g, so that the symbol of the operator $g \mapsto \delta^* v$ is the same as that of $-2\delta^*\delta(h - \frac{1}{2}\mathrm{Tr}\ h)$. Thus the r.h.s. of the equation

$$\frac{dg}{dt} = -2\,\mathrm{Ric} - 2\delta^* v$$

has the same symbol as the rough Laplacian $\Delta = -\nabla^*\nabla = \nabla^2_{e_i, e_i}$, and it is strict parabolic and we get a solution. Then $\phi_t^* g$ satisfies the Ricci flow equation,

$$\frac{d(\phi_t^* g)}{dt} = \phi_t^*\Big(-2\,\mathrm{Ric} - 2\delta^* v\Big) + 2\phi_t^*(\delta^* v) = -2\,\mathrm{Ric}(\phi^* g).$$

For this v, DeTurck used $v = T^{-1}\delta(T - (\mathrm{Tr}\ T)g/2)$, where T is any *fixed* nondegenerate symmetric 2-tensor; T^{-1} is defined by $T((T^{-1}w)^\sharp, X) = w(X)$, for all 1-tensors w, in local coordinate

$$T^{-1} : w_i dx_i \mapsto T^{pq} w_q g_{pi} dx_i.$$

We now compute the symbol of $g \mapsto \delta^* v$. As before, take a variation $g + sh$. Below every equation is mod 0th and 1st derivatives of h. We start from

$$(\delta^* T^{-1}\delta T)'_s = \delta^* T^{-1}(\frac{d}{dt}\delta T) + ...$$

We compute, using an evolving frame $e_i(s)$,

$$\frac{d}{dt}\delta T(X) = -(\nabla T)'(e_i, e_i, X) + ... = T(\nabla'_{e_i} e_i, X) + T(e_i, \nabla'_{e_i} X) + ...$$

Recall, $2 < \nabla'_X Y, Z >= \nabla_X h(Y,Z) + \nabla_Y h(X,Z) - \nabla_Z h(X,Y)$, so

$$\delta^*(T^{-1}T(\nabla'_{e_i} e_i, \cdot)) = \delta^*(T^{kp}T_{jk}(-(\delta h)_j - \frac{1}{2}(d\mathrm{Tr}h)_j)e^p) = -\delta^*\delta(h - \frac{1}{2}(\mathrm{Tr}h)g).$$

Similarly, write $h_{ij,k} = \nabla_{e_k} h(e_i, e_j)$,

$$\delta^*(T^{-1}T(e_i, \nabla'_{e_i}\cdot)) = \delta^*(T^{pk}(\frac{1}{2}h_{kj,i} + \frac{1}{2}h_{ij,k} - \frac{1}{2}h_{ik,j})T_{ij}e^p) = \delta^*(\frac{1}{2}h_{ij,k}T^{pk}T_{ij}e^p).$$

Then consider $\delta^* T^{-1}\delta(\mathrm{Tr}T)g$,

$$(\delta^* T^{-1}\delta(\mathrm{Tr}T)g)'_s = \delta^* T^{-1}(\frac{d}{dt}\delta)(\mathrm{Tr}T)g + \delta^* T^{-1}\delta(\frac{d}{dt}\mathrm{Tr}T)g + \delta^* T^{-1}\delta(\mathrm{Tr}T)h + ...$$

The first part is, mod 0th and 1st derivatives of h,

$$\delta^* T^{-1}(\mathrm{Tr}T < \nabla'_{e_i} e_i, \cdot > + \mathrm{Tr}T < e_i, \nabla'_{e_i}\cdot >) = -\delta^* T^{-1}\mathrm{Tr}T\delta h,$$

it cancels the last. The second part is $-\delta^* T^{-1}\delta(< T, h > g)$. So, mod 0th and 1st derivatives of h,

$$(\delta^* T^{-1}\delta(\mathrm{Tr}T)g)'_s \sim -\,\delta^* T^{-1}\delta(< T, h > g) = \delta^*(T^{ij}T_{pq}h_{pq,i}e^j).$$

$\square$

We now study a family of metric $g(t)$ (want it to be a solution of Ricci flow). Assume $g_0(t)$ is a solution to the Ricci flow equation.

Instead of a fixed T, we can choose a tensor T that depends on time. Following [2], we pick a special vector field V defined by

$$V(t) = \mathrm{Tr}^t(\nabla^t - \nabla).$$

Here ∇ is the connection of a *fixed background metric* $g_0(t) = T(t)$. Note $g_0(t)$ changes with respect to time. ∇^t is the connection of our metric $g(t)$. Note the trace is taken for $g(t)$.

Let W be its dual form, so

$$W_i = g_{ik}(t)g^{pq}(t)(\Gamma^k_{pq}(g(t)) - \Gamma^k_{pq}(g_0(t))).$$

Recall $T = g_0(t)$, one computes $W = T^{-1}\delta(T - (\mathrm{Tr}\ T)g/2))$: check directly,

$$\begin{aligned} W(t)_l = & -\frac{1}{2} g(t)_{lk} g_0(t)^{kp} g(t)^{ij} (\nabla_i^t (g_0(t))_{jp} + \nabla_j^t (g_0(t))_{ip} - \nabla_p^t (g_0(t))_{ij}) \\ = & g(t)_{lk} g_0(t)^{kp} \delta(g_0 - (\mathrm{Tr}\ {}^t g_0(t)) g(t)/2). \end{aligned}$$

So instead of solving Ricci flow directly, one solves

$$\frac{d}{dt} g_{ij} = -2 \,\mathrm{Ric}_{ij} + \nabla_i W_j + \nabla_j W_i, \tag{31.8}$$

which is parabolic. In fact, one can similarly compute the linearized equations as

$$\frac{d}{dt} h_{ij} = \Delta h_{ij} + 2R_{kijl} h_{kl} - R_{ik} h_{kj} - R_{jk} h_{ki}.$$

Finally one runs the diffeomorphism to recover the Ricci flow. See [2] section 2 for details.

References

[1] *Collected Papers on Ricci Flow.* Edited by H.D. Cao, B. Chow, S.C. Chu, and S.T. Yau. Contains 24 articles by Richard S. Hamilton, Dennis DeTurck, Gerhard Huisken, Bennett Chow, S.C. Chu, Richard Schoen, William Meeks, Leon Simon, S.T. Yau, B.L. Chen and X.P. Zhu. International Press, 2002.

[2] G. Anderson, B. Chow, *A pinching estimate for solutions of the linearized Ricci flow system on* 3*-manifolds.* preprint.

[3] Y. Burago, M. Gromov, G. Perelman, *A. D. Alexandrov spaces with curvatures bounded below.* Russian Math. Surveys 47 (1992), 1-58.

[4] J. Cheeger, T. Colding, *On the structure of spaces with Ricci curvature bounded below, I.* J. Diff. Geom. 45 (1997), 406-480.

[5] B. Chow, D. Knopf, *The Ricci flow, vol 1.*

[6] T. Colding, W. Minicozzi, *Estimates for the extinction time for the Ricci flow on certain 3-manifolds and a question of Perelman.* preprint.

[7] K. Fukaya, T. Yamaguchi, *The fundamental groups of almost nonnegatively curved manifolds.* Ann. Math. 136 (1992) 253-333

[8] R. Hamilton, *Three -manifolds with positive Ricci curvature.* J. Diff. Geom. 1982.

[9] R. Hamilton, *Four-manifolds with positive curvature operator.* J. Diff. Geom. 1986.

[10] R. Hamilton, *A compactness property for solutions of the Ricci flow.* Amer. J. Math., 117 (1995) 545-572.

[11] R. Hamilton, *The formation of singularities in the Ricci flow.* Surveys in Differential Geometry II.

[12] R. Hamilton, *Nonsingular solutions of the Ricci flow on three-manifolds.* Comm. Anal. Geom. 1999.

[13] T. Ivey, *Ricci solitons on compact three-manifolds.* Differential Geom. Appl. 3 (1993), no. 4, 301–307

[14] B. Kleiner, J. Lott, *Notes on Perelman's paper.*
http://www.math.lsa.umich.edu/research/ricciflow/perelman.html

[15] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications.* preprint.

[16] G. Perelman *Ricci flow with surgery on three-manifolds.* preprint.

[17] G. Perelman *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds.* preprint.

[18] W. X. Shi, *Deforming the metric on complete Riemannian manifolds.* J. Diff. Geom. 30 (1989) 223-301.

[19] W. X. Shi, *Ricci Deformation of the metric on complete noncompact Riemannian manifolds.* J. Diff. Geom. 30 (1989) 303-394.

[20] T. Shioya, *Behavior of distant maximal geodesics in finitely connected complete* 2*-dimensional Riemannian manifolds.* Mem. AMS. 108 (1994). No. 517.

[21] T. Shioya, T. Yamaguchi, *Collapsing three-manifolds under a lower curvature bound.* J. Diff. Geom. 56 (2000) 1-66.

[22] T. Shioya, T. Yamaguchi, *Volume collapsed three-manifolds with a lower curvature bound.* ArXiv-DG0304472

[23] R. Schoen, S-T. Yau, *Lectures on Differential Geometry* International Press, 1995.

[24] T. Yamaguchi, *Collapsing and pinching under a lower curvature bound.* Ann. Math. 133 (1991), 317-357.

[25] T. Yamaguchi, *A convergence theorem in the geometry of Alexandrov spaces.* Actes de la Table Ronde de Géométrie Différentielle (Luminy, 1992), Sémin. Congr., Vol. 1, Soc. Math. France, Paris, 1996, 601-642.

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