# HEAT KERNELS AND GREEN'S FUNCTIONS ON LIMIT SPACES

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Abstract. In this paper, we study the behavior of the Laplacian on a sequence of manifolds $\{M_i^n\}$ with a lower bound in Ricci curvature that converges to a metric-measure space $M_\infty$. We prove that the heat kernels and Green's functions on $M_i^n$ will converge to some integral kernels on $M_\infty$ which can be interpreted, in different cases, as the heat kernel and Green's function on $M_\infty$. We also study the Laplacian on noncollapsed metric cones; these provide a unified treatment of the asymptotic behavior of heat kernels and Green's functions on noncompact manifolds with nonnegative Ricci curvature and Euclidean volume growth. In particular, we get a unified proof of the asymptotic formulae of Colding-Minicozzi, Li and Li-Tam-Wang.

## Introduction

Assume $M^n$ is an $n$ dimensional Riemannian manifold with a lower bound in Ricci curvature,

$$\mathrm{Ric}_{M^n} \geq -(n-1)\Lambda, \tag{0.1}$$

where $\Lambda \geq 0$. By the Bishop-Gromov inequality, we have a *uniform* volume doubling condition,

$$\mathrm{Vol}(B_{2R}(p)) \leq 2^\kappa \,\mathrm{Vol}(B_R(p)), \tag{0.2}$$

here we can take $\kappa = n$ if $\Lambda = 0$; if $\Lambda > 0$, we require that $R$ is bounded from above, say, $R < D$ for some $D > 0$. Moreover, there is a *uniform* Poincare inequality

$$\frac{1}{\mathrm{Vol}(B_R(p))}\int_{B_R(p)} |f - f_{p,R}| \leq \tau R \Big(\frac{1}{\mathrm{Vol}(B_R(p))}\int_{B_R(p)} |df|^2\Big)^{\frac{1}{2}}, \tag{0.3}$$

where $f_{p,R}$ is the average of $f$ on $B_R(p)$; see [Bu], [Ch3], [HaKo] and the references therein.

We assume, throughout this paper, $\{M_i^n\}$ is a sequence of complete Riemannian manifolds with (0.1) that converges in the pointed measured Gromov-Hausdorff sense, to a metric space $M_\infty$; we write $M_i^n \xrightarrow{d_{GH}} M_\infty$, $d_{GH}$ is the Gromov-Hausdorff distance. In particular, (0.2) holds on $M_\infty$. One can show that (0.3) and the *segment inequality*, which is stronger than (0.3), hold on $M_\infty$ as well; note on $M_\infty$, the role of $|du|$ in (0.3) is played by $g_f$, the *minimal generalized upper gradient*, see [Ch3], [ChCo4].

In Cheeger's paper [Ch3], a significant portion of analysis on smooth manifolds was extended to metric-measure spaces satisfying (0.2), (0.3). In [Ch3], [ChCo4] Cheeger and Colding defined a self-adjoint Laplacian operator $\Delta$ on $M_\infty$. By convention $\Delta$ is *positive*. They proved that the eigenvalues and eigenfunctions of the Laplacian $\Delta_i$ over $M_i^n$ converge to those on $M_\infty$, thereby establishing Fukaya's conjecture [Fu]. So

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if we consider $\mathcal{RIC}$, the completion under measured Gromov-Hausdorff convergence of the set of smooth manifolds with (0.1), it is natural to expect that quantities associated to $\Delta$ should behave continuously.

In this paper, we will study this phenomenon in detail. Our main goal is to prove, in various cases, that the heat kernel $H_i$ and Green's function $G_i$ on $M_i^n$ converge uniformly to the heat kernel $H_\infty$ and Green's function $G_\infty$ on $M_\infty$. We will make precise the definition of these convergences in Section 1. For results concerning heat kernels, a lower bound in Ricci curvature (0.1) is enough; for Green's functions, we require that $\mathrm{Ric}_{M_i^n} \geq 0$; compare with (1.16), (1.18), (1.19).

Moreover, in the noncompact case, we also study the asymptotics of the heat kernel and Green's function on a manifold $M^n$ with $\mathrm{Ric}_{M^n} \geq 0$ and a Euclidean volume growth condition:

$$\mathrm{Vol}(B_R(p)) > v_0 R^n. \tag{0.4}$$

According to [ChCo1], any tangent cone at infinity of a manifold $M^n$ with $\mathrm{Ric}_{M^n} \geq 0$ and (0.4) is a metric cone $C(X)$. So viewed from a sufficiently large scale, $M^n$ appears to be close to some $C(X)$. Combined with the appropriate convergence theorems mentioned above, at a sufficiently large scale the heat kernel and Green's function on $M^n$ are close to those on $C(X)$.

On the other hand, we show that the classical analysis on cones, [Ch1], [Ch2], [ChTa1], can be generalized to $C(X)$. In particular, we have explicit expression of heat kernels and Green's functions on $C(X)$; see (6.21), (4.23). In this way we get a unified treatment for the asymptotic formulae of these integral kernels on $M^n$. In particular, we get new proofs of the Colding-Minicozzi asymptotic formula for Green's functions, [CoMi1] (compare with [LiTW]), the asymptotic formulae for heat kernels of Li [Li1] and Li-Tam-Wang [LiTW].

The organization of this paper is as follows:

Section 1 reviews some background material that we need in the sequel.

In Section 2, in the *compact* case we prove, $H_i(\cdot,\cdot,t) \to H_\infty(\cdot,\cdot,t)$ uniformly (assuming (0.1)), and $G_i \to G_\infty$ uniformly, off the diagonal (assuming $\mathrm{Ric}_{M^n} \geq 0$). It's well known that there is an eigenfunction expansion for heat kernels, so our results follows easily from the work of Cheeger-Colding [ChCo4], [Ch3], by estimating the remainders of the eigenvalue expansions. We remark, previously in [KK1], [KK2] it was proved that a *subsequence* of $H_i$ converges to *some* kernel on the compact metric space $M_\infty$.

By the Dirichlet's principle and the transplantation theorem of Cheeger [Ch3], we show in Section 3 that the uniform limit of solutions for Poisson equations is a solution for a Poisson equation, see also [Ch3], [ChCo4]. In particular, if $\{M_i^n\}$ are noncompact, $\mathrm{Ric}_{M_i^n} \geq 0$, satisfy (0.4) uniformly, then $G_i \to G_\infty$ uniformly off the diagonal (Theorem 3.21).

We treat the heat kernels on noncompact spaces in Section 5. We assume (0.1). First, some *subsequence* of the Dirichlet heat kernels $H_{R,i}$ on $B_R(p_i) \subset M_i^n$ will converge to some function $H_{R,\infty}$ on $B_R(p_\infty) \subset M_\infty$. However, at present it is not clear if $H_{R,i}$ will converge. On the other hand by a generalized maximum principle, any two $H_{R,\infty}$ (from two different subsequences) can not be too different from each other, see (5.46). Letting $R \to \infty$, we prove that $H_i(\cdot,\cdot,t) \to H_\infty(\cdot,\cdot,t)$ in $L^1$.

In Theorem 5.59, when the noncollapsed limit $M_\infty$ is a manifold, we prove $H_\infty$ is the heat kernel over $M_\infty$, i.e. the integral kernel of the semigroup $e^{-t\Delta}$. For general $M_\infty$, the picture is not yet clear; however, it is true when $M_\infty = C(X)$ is a noncollapsed tangent cone that is the limit of a sequence of manifolds with nonnegative Ricci curvature, see Theorem 6.1.

In Section 4 and Section 6 we study the Laplacian on $C(X)$. We prove that in this case, one can still separate variables. We use these to study the structure of $G_\infty$ and $H_\infty$ on $C(X)$, and the asymptotic behavior of Green's function and heat kernel on a manifold $M^n$ with $\mathrm{Ric}_{M^n} \geq 0$ and (0.4).

In Section 7 we study the asymptotic behavior of the eigenvalues $\lambda_{j,\infty}$ on a compact metric space $M_\infty$ which is the limit of a sequence of manifolds $\{M_i^n\}$ with (0.1). We will prove in the noncollapsed case, the Weyl asymptotic formula is true on $M_\infty$; see Theorem 7.3. In the collapsed case, we get some link between the behavior of eigenvalues and $\dim_{Mink}(M_\infty)$, the Minkowski dimension of $M_\infty$.

All of the estimates in this paper are *uniform*, i.e. the constants are valid for the whole family of manifolds we are considering (for example all compact manifolds $M^n$ with $\mathrm{Ric}_{M^n} \geq -(n-1)\Lambda$ and $\mathrm{Diam}\, M^n \leq D$).

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## 1. Background and notation

Suppose $(M_i^n, \mathrm{Vol}_i) \xrightarrow{d_{GH}} (M_\infty, \mu_\infty)$ in the measured Gromov-Hausdorff sense, i.e. the sequence $\{M_i^n\}$ converges in the Gromov-Hausdorff sense to $M_\infty$, and for any $x_i \to x_\infty$, ($x_i \in M_i^n$) and $R > 0$, we have $\mathrm{Vol}_i(B_R(x_i)) \to \mu_\infty(B_R(x_\infty))$; here $\mu_\infty$ is Borel regular. In fact, for any sequences of manifolds with Ricci curvature bounded from below, after possible renormalization of the measures when $\{M_i^n\}$ is collapsing, one can alway find a subsequence converges in the measured Gromov-Hausdorff sense, see [ChCo2]. In the following we usually let Vol denote the (renormalized if $\{M_i^n\}$ is collapsing) measure on $M_i^n$, $i = 1, 2, ..., \infty$; on $M_\infty$ sometimes we also write $\mu_\infty$ for $\mathrm{Vol}_\infty$. We refer to [Ch3], [ChCo2], [Gr] for general background on measured Gromov-Hausdorff convergence.

**Definition 1.1.** *Suppose $K_i \subset M_i^n \xrightarrow{d_{GH}} K_\infty \subset M_\infty$ in the measured Gromov-Hausdorff sense. $f_i$ is a function on $M_i^n$, $i = 1, 2, ...$ ; $f_\infty$ is a continuous function on $M_\infty$. Assume $\Phi_i : K_\infty \to K_i$ are $\epsilon_i$-Gromov-Hausdorff approximations, $\epsilon_i \to 0$. If $f_i \circ \Phi_i$ converge to $f_\infty$ uniformly, we say that $f_i \to f_\infty$* uniformly *over $K_i \xrightarrow{d_{GH}} K_\infty$.*

For simplicity, in the above context, we also say that $f_i \to f_\infty$ uniformly on $K$; when we write $f_i(x) \to f_\infty(x)$, we mean that $f_i \to f_\infty$ uniformly and $f_i(x_i) \to f_\infty(x_\infty)$, where $x_i \to x_\infty$, $x_i \in M_i$.

In many applications, the family $\{f_i\}$ is actually equicontinuous. We remark, the Arzela-Ascoli theorem can be generalized to the case where the functions live on different spaces: when $M_i^n \xrightarrow{d_{GH}} M_\infty$, for any bounded, equicontinuous sequence $\{f_i\}$

($f_i$ is a function on $M_i^n$), there is a subsequence that converges uniformly to some continuous function $f_\infty$ on $M_\infty$. The proof is straightforward.

We also introduce the notion of $L^p$ *convergence* ($1 \le p \le \infty$).

**Definition 1.2.** *We say $f_i \to f_\infty$ in $L^p$, if for all $\epsilon > 0$, one can write $f_i = \phi_i + \eta_i$ such that $\phi_i \to \phi_\infty$ uniformly and* $\limsup_{i\to\infty} \|\eta_i\|_{L^p} \le \epsilon$, $\|\eta_\infty\|_{L^p} \le \epsilon$.

The following is a generalization of the Rellich-Kondrakov theorem:

**Lemma 1.3** (Rellich)**.** *Assume $B_1(p_i) \subset M_i^n \xrightarrow{d_{GH}} B_1(p) \subset M_\infty$ in the measured Gromov-Hausdorff sense. $u_i$ is a function on $M_i^n$, $i = 1, 2, \ldots$ . Assume*

$$\int_{B_1(p_i)} (u_i)^2 + |\nabla u_i|^2 \le N. \tag{1.4}$$

*Then there is a subsequence of $\{u_i\}$ that converges in $L^2$ over any compact subset of the open balls $B_1$.*

The proof depends only on a *weak Poincare inequality* (use a bigger ball on the right of (0.3)) and (0.2). One can divide the ball $B_R(p)$ ($R < 1$) into small subsets and approximate $f$ by functions that are constant over each of these small subsets, then one easily finds a convergent subsequence by standard diagonal arguments. Compare [Ch3], especially [CoMi3].

We use subscript $i$ and write $f_i, H_{R,i}, p_i$, etc. to denote functions, points, etc on $M_i^n$. To simplify notation, when we write an equation with some function or other objects with no subscription (for example, $f$), it should be understood that the equation is valid for some suitable convergent sequence of functions or other objects (for example, $\{f_i\}$; $f_i$ is defined on $M_i^n$, $i = 1, 2, \ldots$), according to the context.

In [Ch3], Cheeger defined a Sobolev space $H_{1,2}$ on metric-measure spaces $(Z, \mu)$ satisfying (0.2), (0.3), and proved that Lipschitz functions are dense in $H_{1,2}$. Denote by $\overset{\circ}{H}_{1,2}(\Omega)$ the closure in $H_{1,2}$ of Lipschitz functions supported in an open set $\Omega$. Recall in [Ch3], one has a natural finite dimensional cotangent bundle $T^*Z$. We use $du$ to denote the *differential* of $u$, see Section 4 of [Ch3]. One can put a norm $|\cdot|$ on $T^*Z$ by assigning $|df| = g_f = \mathrm{Lip}\, f$ for $f$ Lipschitz. Here as in [Ch3],

$$\mathrm{Lip}\, f(x) = \limsup_{y\to x} \frac{|f(y) - f(x)|}{d(y, x)}. \tag{1.5}$$

Note we use $\mathbf{Lip} f$ to denote the Lipschitz constant of $f$. Clearly, on smooth manifold $|\cdot|$ agrees with the standard norm $|du| = |\nabla u|$. It was proved in [Ch3] that $|\cdot|$ is equivalent to a uniformly convex norm, in particular, an inner product.

If we have the stronger assumption $Z = M_\infty$ with $M_i^n \xrightarrow{d_{GH}} M_\infty$, in [ChCo4] Cheeger and Colding proved that $M_\infty$ is $\mu_\infty$-rectifiable, and, as a corollary, the norm $|\cdot|$ actually comes from an inner product $< \cdot, \cdot >$. So $H_{1,2}$ is made into a Hilbert space:

$$< u, v >_{H_{1,2}} = \int_{M_\infty} uv + \int_{M_\infty} < du, dv >, \tag{1.6}$$

and for Lipschitz functions $f$, one has

$$\|f\|^2_{H_{1,2}} = \|f\|^2_{L^2} + \int_{M_\infty} |\,\mathrm{Lip}\, f|^2. \tag{1.7}$$

Now by the standard theory of Dirichlet forms, one gets a *positive* self-adjoint Laplacian $\Delta$ on $M_\infty$, see [Ch3], [ChCo4] for the details of this theory.

Recall one form of the transplantation theorem of Cheeger (for a proof, see Lemma 10.7 of [Ch3]):

**Lemma 1.8.** *Assume $M_i^n \xrightarrow{d_{GH}} M_\infty$. $f_\infty$ is a Lipschitz function on $B_R(x_\infty) \subset M_\infty$, $x_i \to x_\infty$. Then there is a sequence of Lipschitz functions $\{f_i\}$ that converges uniformly to $f_\infty$, here $f_i$ is defined on $B_R(x_i) \subset M_i^n$. Moreover, one can require that*

$$\limsup_{i\to\infty} \mathbf{Lip} f_i \leq \mathbf{Lip} f_\infty, \tag{1.9}$$

$$\limsup_{i\to\infty} \| \operatorname{Lip} f_i \|_{L^2} \leq \| \operatorname{Lip} f_\infty \|_{L^2}. \tag{1.10}$$

By [J], [HaKo], on length spaces satisfying (0.2), a weak Poincare inequality implies a uniform Poincare-Sobolev inequality (i.e, put $L^{\chi p}$ norm on the left side of (0.3) for some $\chi > 1$). In particular, we have a Dirichlet-Sobolev inequality for $u \in \overset{\circ}{H}_{1,2}(B_R(p))$. So we have

**Lemma 1.11** (Moser iteration)**.** *If for all $\phi$ with compact support in $B_2(p)$,*

$$\int \nabla u \nabla \phi = \int cu\phi + f\phi, \tag{1.12}$$

*then for $q > C(\kappa)$, we have,*

$$\|u\|_{L^\infty(B_1(p))} \leq C(n,q)(1+|c|)^{N(\tau,\kappa)}(\|u\|_{L^2(B_2(p))} + \|f\|_{L^q(B_2(p))}). \tag{1.13}$$

For proof, see chapter 4 of [Lin]. Note here we need to renormalize the measure.

Recall, on smooth manifolds we have

**Lemma 1.14** (Gradient estimate)**.** *If $\Delta u - cu = f$, $\|u\|_{L^2} < \infty$ on $B_{2R}(p)$ and $f$ is a $C^2$ function with Lipschitz constant $\mathbf{Lip} f$, $c$ is a constant, then on $B_R(p)$ we have a gradient estimate:*

$$|\nabla u| \leq C(\|f\|_{L^\infty}, \|u\|_{L^2}, \mathbf{Lip} f)(1+|c|)^{N(\tau,\kappa)}. \tag{1.15}$$

The proof follows a standard argument of Cheng-Yau, see [CY1], [LiY1], compare also with [Lin], [Li2], [SY]. Then use the Moser iteration to replace $\|u\|_{L^\infty}$ with $\|u\|_{L^2}$.

Finally recall the Li-Yau estimates [LiY2], [SY]: If $M^n$ satisfies (0.1), then its heat kernel $H$ satisfies

$$H(x,y,t) \leq C(n) \operatorname{Vol}(B_{\sqrt{t}}(x))^{-1/2} \operatorname{Vol}(B_{\sqrt{t}}(y))^{-1/2} e^{-d^2(x,y)/5t} e^{C\Lambda t}. \tag{1.16}$$

If $t < T$, by volume comparison (1.16) simplifies to

$$H(x,y,t) \leq C(n,\Lambda,T) \operatorname{Vol}(B_{\sqrt{t}}(x))^{-1} e^{-d^2(x,y)/5t} e^{C\Lambda t} t^{-C(n)} e^{C(n,\Lambda,T)d(x,y)}. \tag{1.17}$$

If we assume $\operatorname{Ric}_{M^n} \geq 0$, then

$$\frac{C^{-1}(n)}{\operatorname{Vol}(B_{\sqrt{t}}(x))} e^{-d^2(x,y)/3t} \leq H(x,y,t) \leq \frac{C(n)}{\operatorname{Vol}(B_{\sqrt{t}}(x))} e^{-d^2(x,y)/5t}. \tag{1.18}$$

Assume $M^n$ is noncompact, write $G$ for the minimal positive Green's function on $M^n$. If $n \geq 3$, $M^n$ satisfies (0.4) and $\mathrm{Ric}_{M^n} \geq 0$, then $G$ exists ([LiY2], [LiT1], [LiT2], [SY]) and satisfies

$$C^{-1}(v_0)d(x,y)^{2-n} \leq G(x,y) \leq C(v_0)d(x,y)^{2-n}. \tag{1.19}$$

For proofs of the above estimates, see [LiY2], [SY].

## 2. The compact case

In this section, we assume $M_\infty$ is compact. First we study the heat kernel $H$.

It's well known that if $\mathrm{Ric}_{M_i^n} \geq -(n-1)\Lambda$, there is a lower bound for the $k$th eigenvalue $\lambda_{k,i}$ of the Laplacian $\Delta_i$ over $M_i^n$:

$$\lambda_{k,i} \geq C(n,\Lambda,D)k^{\frac{2}{n}}, \tag{2.1}$$

here $\mathrm{Diam}\, M_i^n \leq D$. The proof uses only (0.2) and (0.3); see [Gr] and Theorem 4.8 of [Ch3]. Hence over $M_i^n$ we have

$$H_i(x,y,t) = \sum_{j=0}^{\infty} e^{-\lambda_{j,i}t}\phi_{j,i}(x)\phi_{j,i}(y), \tag{2.2}$$

here $\phi_{j,i}$ is the eigenfunction of the $j$th eigenvalue $\lambda_{j,i}$. By the Cheeger-Colding spectral convergence theorem [ChCo4], for each $j$, $\lambda_{j,i} \to \lambda_{j,\infty}$, and $\phi_{j,i} \to \phi_{j,\infty}$ uniformly when $i \to \infty$, here $\lambda_{j,\infty}$ and $\phi_{j,\infty}$ are the $j$-th eigenvalue and (renormalized) eigenfunction of $\Delta$ on $M_\infty$. So (2.1) is also true for $\lambda_{j,\infty}$. Moreover,

$$\|\phi_{j,i}\|_{L^\infty} \leq C_1(n,\Lambda,D)(1+\lambda_{j,i})^{C(n)}\|\phi_{j,i}\|_{L^2}, \tag{2.3}$$

$$\|\nabla\phi_{j,i}\|_{L^\infty} \leq C_0(n,\Lambda,D)(1+\lambda_{j,i})^{C(n)}\|\phi_{j,i}\|_{L^\infty}. \tag{2.4}$$

These are implied by (the proof of) the Moser iteration and the gradient estimate, see [LiY1]. By [ChCo4], (2.3), (2.4) can pass to $M_\infty$ (on $M_\infty$ (2.4) becomes an estimate for $\mathbf{Lip}\phi_{j,\infty}$). So it makes sense to write

$$H_\infty(x,y,t) = \sum_{j=0}^{\infty} e^{-\lambda_{j,\infty}t}\phi_{j,\infty}(x)\phi_{j,\infty}(y). \tag{2.5}$$

By (2.1), $H_\infty$ is the heat kernel over $M_\infty$.

Apply (2.3), (2.4) to $\sum_{j=k}^{\infty} e^{-\lambda_{j,i}t}\phi_{j,i}(x)\phi_{j,i}(y)$, the tail of (2.2), one easily get

**Theorem 2.6.** *Assume $M_i^n \xrightarrow{d_{GH}} M_\infty$, $\mathrm{Ric}_{M_i^n} \geq -(n-1)\Lambda$. When $t > 0$ fixed, $H_i$ converges to the heat kernel $H_\infty$ over $M_\infty$ uniformly. $H_\infty$ is continuous in $t, x, y$; when $t$ fixed, it is Lipschitz in $x, y$.*

**Corollary 2.7.** *For $H_\infty$ on $M_\infty$, the Li-Yau estimate (1.18) is true if $\mathrm{Ric}_{M_i^n} \geq 0$; if $\mathrm{Ric}_{M_i^n} \geq -(n-1)\Lambda$ then (1.16) is true.*

Next we study the Green's functions. Assume $\mathrm{Ric}_{M_i^n} \geq 0$. Recall,

$$G_i(x,y) = \int_0^\infty h_i(x,y,t)dt, \tag{2.8}$$

here $h_i(x,y,t) = H_i(x,y,t) - \phi_{0,i}(x)\phi_{0,i}(y) = H_i(x,y,t) - 1$. Note, since the sequence $\{M_i^n\}$ might collapse, we have to renormalize the measures and eigenvalues such that $\mathrm{Vol}(M_i^n) = 1$ and $\{\phi_{j,i}\}_j$ is orthonormal, e.g. $\lambda_{0,i} = 0$, $\phi_{0,i} = 1$. So

$$G_i(x,y) = \int_0^{\epsilon} h_i(x,y,t)dt + \sum_{j=1}^{\infty} e^{-\epsilon\lambda_{j,i}/2} \int_{\frac{\epsilon}{2}}^{\infty} e^{-\lambda_{j,i}t}\phi_{j,i}(x)\phi_{j,i}(y)dt. \tag{2.9}$$

When $x$ is fixed, by (2.3),

$$\sum_{j=k}^{\infty} \| e^{-\epsilon\lambda_{j,i}/2} \int_{\frac{\epsilon}{2}}^{\infty} e^{-\lambda_{j,i}t}\phi_{j,i}(x)\phi_{j,i}(y)dt\|_{L^\infty} \leq \sum_{j=k}^{\infty} \frac{e^{-\epsilon\lambda_{j,i}}}{\lambda_{j,i}} C_1(1+\lambda_{j,i})^{C_3}. \tag{2.10}$$

This goes to 0 uniformly in $i$ as $k \to \infty$, by (2.1). On the other hand, clearly (1.18) holds after renormalization; when $\mathrm{Ric} \geq 0$, $R < 1/8$ we have the (rescaled) volume bound

$$C(n,D)R^n \leq \mathrm{Vol}(B_R(x)) \leq \sqrt{R}C(n)\,\mathrm{Vol}(B_{\sqrt{R}}(x)) \leq C'(n)R. \tag{2.11}$$

So when $d(x,y) \geq \delta > 0$, by (1.18),

$$\int_0^{\epsilon} |h_i(x,y,t)|dt \leq \epsilon + \int_0^{\epsilon} |H_i(x,y,t)|dt \leq \epsilon + C\int_0^{\epsilon} t^{-\frac{n}{2}} e^{-\frac{\delta^2}{5t}}dt. \tag{2.12}$$

So in particular, by choosing $\epsilon$ small, we get a function $G_\infty(x,y)$ on $M_\infty$, such that $G_i \to G_\infty$ in $L^\infty$ on compact subsets, off the diagonal.

Finally, we want to check $G_\infty$ is the Green's function over $M_\infty$. We now establish an $L^1$ bound for $G(x,y)$ over the ball $B_R(x)$. Note

$$\int_{B_R(x)} |G(x,y)|dy \leq \int_{B_R(x)}\int_0^1 |h(x,y,t)|dtdy + \int_{B_R(x)}\int_1^{\infty} |h(x,y,t)|dtdy. \tag{2.13}$$

Since $\|\phi(y)\|_{L^2} = 1$, by (2.1), (2.3), (2.11) and the Schwartz inequality,

$$\begin{aligned} \int_{B_R(x)}\int_1^{\infty} |h(x,y,t)|dtdy &\leq \sum_{j=1}^{\infty}\int_{B_R(x)} e^{-\lambda_j/2}|\phi_j(x)| \int_{\frac{1}{2}}^{\infty} e^{-\lambda_j t}|\phi_j(y)|dtdy \\ &\leq C(n)\sum_{j=1}^{\infty} e^{-\lambda_j/2}(1+\lambda_j)^{C_2(n)}\sqrt{R}\int_{\frac{1}{2}}^{\infty} e^{-\lambda_j t}dt \leq C'(n)\sqrt{R}. \end{aligned} \tag{2.14}$$

Now we focus on the first term on the right hand side of (2.13). Since $H - h = 1$, and we have (2.11), it's enough to estimate the integral of $H$. Put $R < 1/8$, by (1.18), (2.11) we have

$$\begin{aligned} \int_0^1\int_{B_R(x)} H(x,y,t)dtdy &\leq \int_0^1\int_{B_R(x)} C(n)\,\mathrm{Vol}^{-1}(B_{\sqrt{t}}(x))e^{-\frac{d(x,y)^2}{5t}}dtdy \\ &\leq (\int_0^R\int_{B_R(x)} C(n)\,\mathrm{Vol}^{-1}(B_{\sqrt{t}}(x))e^{-\frac{d(x,y)^2}{5t}}dtdy) + C'(n)\sqrt{R}. \end{aligned} \tag{2.15}$$

Next,

$$\int_0^R \frac{\int_{B_R(x)} e^{-d(x,y)^2/5t}dy}{\mathrm{Vol}(B_{\sqrt{t}}(x))}dt = \int_0^R \frac{\int_0^R e^{-r^2/5t}A(r)r^{n-1}dr}{\int_0^{\sqrt{t}} A(r)r^{n-1}dr}dt, \tag{2.16}$$

here $A(r)r^{n-1}$ is the surface area element of $\partial B_r(x)$. Since $\mathrm{Ric}_{M_i^n} \geq 0$, $A(r)$ is non-increasing. The right hand side of (2.16) can be bounded by

$$\begin{aligned}\int_0^R \Big( \sum_{s=0}^{[R/\sqrt{t}]+1} \int_{s\sqrt{t}}^{(s+1)\sqrt{t}} e^{-r^2/5t} A(r) r^{n-1} dr \Big/ \int_0^{\sqrt{t}} A(r) r^{n-1} dr\Big) dt \\ \leq C_1(n)R + \int_0^R \sum_{s=1}^{[R/\sqrt{t}]+1} e^{-s^2/5}((s+1)^n - s^n) dt \leq C'(n)R.\end{aligned} \tag{2.17}$$

So combine (2.14), (2.15) and (2.11) we get

$$\int_{B_R(x)} |G(x,y)| dy \leq C'(n)\sqrt{R}. \tag{2.18}$$

Since $G_i \to G_\infty$ uniformly off the diagonal, use the Cheeger-Colding theorem on the convergence of eigenfunctions [ChCo4] and (2.3), (2.4), we get, for all $x$,

$$\begin{aligned}\phi_{j,\infty}(x) = \lim_{i\to\infty} \phi_{j,i}(x) = \lim_{i\to\infty} \int_{M_i} G_i(x,y)\lambda_{j,i}\phi_{j,i}(y) dy \\ = \int_{M_\infty} G_\infty(x,y)\lambda_{j,\infty}\phi_{j,\infty}(y) dy.\end{aligned} \tag{2.19}$$

So $G_\infty$ is the Green's function over $M_\infty$. Moreover, by (2.10), (2.12) and Lemma 1.14, $G_\infty$ is Lipschitz continuous off the diagonal. It is harmonic off the diagonal by Lemma 3.17. So we have proved

**Theorem 2.20.** *Assume $M_i^n \xrightarrow{d_{GH}} M_\infty$, $\mathrm{Ric}_{M_i^n} \geq 0$. Then the Green's function $G_\infty$ on $M_\infty$ exists. On any compact subsets $K$ off the diagonal, $G_\infty$ is Lipschitz and harmonic, $G_i \to G_\infty$ uniformly on $K$.*

## 3. The Green's functions on noncompact spaces

Recall how on a manifold, one solves the the Poisson equation,

$$\Delta u_R = f, \; u_R|_{\partial B_R(p)} = h, \tag{3.1}$$

for Lipschitz functions $f, h$ on the closed ball $B_R(p)$. By the Dirichlet's principle, $u_R$ is the unique minimizer of the functional

$$I(u) = \int_{B_R(p)} (|du|^2 - fu). \tag{3.2}$$

within the space $\mathcal{E} = h + \overset{\circ}{H}_{1,2}(B_R(p))$, note $\Delta$ is positive by convention.

Assume $M_i^n \xrightarrow{d_{GH}} M_\infty$ in the measured Gromov-Hausdorff sense, $\mathrm{Ric}_{M_i^n} \geq -(n-1)\Lambda$. Recall, by [Ch3], [ChCo4], $\Delta$ is linear on $M_\infty$. So the above variational method is valid also on $M_\infty$.

**Lemma 3.3** (Lower semicontinuity of energy)**.** *Suppose $u_i$, $f_i$ are $C^2$ functions over $M_i^n$, $\Delta u_i = f_i$, $u_i \to u_\infty$, $f_i \to f_\infty$ uniformly over the sequence of converging balls $B_{2R}(p_i) \to B_{2R}(p_\infty)$, and there is a uniform gradient estimate for $u_i$ and $f_i$:*

$$|\nabla u_i|, |\nabla f_i| < L. \tag{3.4}$$

*Then we have*

$$I(u_\infty) \le \liminf_{i\to\infty} I(u_i). \tag{3.5}$$

*Proof.* As in [ChCo1], we can get an integral bound for the Hessian of $f_i$ on the ball $B_1(p_i)$: recall the Bochner formula

$$\frac{1}{2}\Delta(|\nabla f_i|^2) = |\mathrm{Hess}_{f_i}|^2 + <\nabla\Delta f_i, \nabla f_i> + \mathrm{Ric}(\nabla f_i, \nabla f_i). \tag{3.6}$$

Multiply by a cut-off function $\phi$ with $\mathrm{supp}\phi \subset B_r \subset B_1(q_i)$, $\phi|_{B_{r/2}} = 1$, $|\nabla\phi| \le c(n,r)$, $|\Delta\phi| \le c(n,r)$; see Theorem 6.33 of [ChCo1]. Since $f_i$ is harmonic,

$$\frac{1}{2}\phi\Delta(|\nabla f_i|^2) = \phi|\mathrm{Hess}_{f_i}|^2 + \phi\,\mathrm{Ric}(\nabla f_i, \nabla f_i). \tag{3.7}$$

Integrate by parts,

$$\int_{B_r}\frac{1}{2}(|\nabla f_i|^2)\Delta\phi = \int_{B_r}\phi|\mathrm{Hess}_{f_i}|^2 + \int_{B_r}\phi\,\mathrm{Ric}(\nabla f_i, \nabla f_i). \tag{3.8}$$

By assumption, there is a definite lower bound for the last term in (3.8). Note $|\Delta\phi|$ is uniformly bounded by construction, we have a uniform upper bound for $\int_{B_r}\phi|\mathrm{Hess}_{f_i}|^2$. So by Lemma 1.3 we can assume some subsequence of $|\nabla f_i|$ converges to a function $\Gamma$ on $B_R(p_\infty) \subset M_\infty$ in $L^2$. Assume, $x \in \mathcal{R}_k \subset M_\infty$ for some $k$ (all tangent cone at $x$ is $\mathbf{R}^k$), there is some subset $A(x) \subset M_\infty$ such that and $\Gamma$ is continuous on $A(x)$, $x \in A(x)$ is a density point of $A(x)$. By Luzin's theorem and the results in [ChCo2], these properties hold for almost all $x \in M_\infty$. For such $x$, we prove

$$|\mathrm{Lip}\ f_\infty(x)| \le \Gamma(x). \tag{3.9}$$

Clearly, (3.9) implies our lemma.

To prove (3.9), it's enough to prove, for all $\psi > 0$, if $l = d(x,y)$ is sufficiently small, then

$$|f_\infty(x) - f_\infty(y)| \le d(y,x)(\Gamma(x) + 6\psi). \tag{3.10}$$

By the gradient estimate of $f_i$ (so of $f_\infty$), if (3.10) is not true for some $y_0$, then for all $y \in B_{l\psi/L}(y_0)$,

$$|f_\infty(x) - f_\infty(y)| > d(y,x)(\Gamma(x) + 5\psi). \tag{3.11}$$

Pick $x_i, y_{0,i} \in M_i^n$, $x_i \to x$, $y_{0,i} \to y_0$, $d(x_i, y_{0,i}) = l$. Then for $i$ big enough, for all $y_i \in B_{l\psi/L}(y_{0,i})$ and all minimal geodesic $\gamma_i$ connecting $x_i$ and $y_i$,

$$\int_{\gamma_i}|\nabla f_i| \ge d(x_i, y_i)(\Gamma(x) + 4\psi). \tag{3.12}$$

First of all, since $|\nabla f_i|$ is uniformly bounded by $L$, a simple computation shows along every $\gamma_i$ we must have

$$|\nabla f_i| > \Gamma(x) + 2\psi, \tag{3.13}$$

on a subset of $\gamma_i$ which has 1-Hausdorff measure at least $2\psi l/(L - \Gamma(x))$. Put

$$\begin{aligned} T_i = \{v \in T_{x_i} | v = \gamma'(0) &\text{ for some minimal geodesic } \gamma \\ &\text{from } x_i \text{ to } y_i \in B_{l\psi/L}(y_{0,i})\}. \end{aligned} \tag{3.14}$$

We must have

$$H^{n-1}(T_i) > C(n, L, \psi) H^{n-1}(\partial B_1(0)), \tag{3.15}$$

where $H^{n-1}$ is the $(n-1)$-Hausdorff measure over the unit sphere $\partial B_1(0)$ in the tangent space $T_{x_i}$. Combine this with (3.13), when $M_\infty$ is *noncollapsed*, if $l$ is small enough, by the proof of the Bishop-Gromov inequality, for sufficiently big $i$,

$$\frac{\text{Vol }(\{z_i \in B_l(x_i) | |\nabla f_i(z_i)| > \Gamma(x) + 2\psi\})}{\text{Vol }(B_l(x_i))} \geq C(x, n, L, \psi) > 0. \tag{3.16}$$

Now $|\nabla f_i|$ converge to $\Gamma$ in $L^2$, so (3.16) is also true if we substitute $|\nabla f_i|$ in (3.16) by $\Gamma$, $x_i$ by $x$. We get a contradiction to the choice of $x$.

The proof is the same when $M_\infty$ is *collapsed*. We just use the *segment inequality* ([ChCo1], [ChCo4]) to get (3.16) from (3.11), (3.12) and (3.13). □

**Lemma 3.17.** *Let $u_\infty$, $f_\infty$ be as in the previous lemma. Then*

$$\Delta u_\infty = f_\infty. \tag{3.18}$$

*Proof.* Assume this is not true over a ball $B_\lambda(p^*) \subset\subset B_1(0)$. By solving the Dirichlet problem on $B_\lambda(p^*)$ we can find $v_\infty$ with the same boundary value as $u_\infty$ over $\partial B_\lambda$, but with smaller energy, say

$$I(v_\infty) = \int_{B_\lambda(p^*)} |dv_\infty|^2 - f_\infty v_\infty < \int_{B_\lambda(p^*)} |du_\infty|^2 - f_\infty u_\infty - 2\Psi. \tag{3.19}$$

By obvious density properties, we can change $v_\infty$ slightly so that $v_\infty$ agrees with $u_\infty$ on a neighborhood of $\partial B_\lambda(p^*)$. By Lemma 3.3, for $i$ big enough,

$$I(v_\infty) \leq I(u_i) - \Psi. \tag{3.20}$$

So by (the proof of) Lemma 1.8 (see Section 10 of [Ch3]), for $i$ big enough we can find a function $v_i$ with the same boundary value on $\partial B_i$ as $u_i$ but with smaller energy $I$. That contradicts the fact that $\Delta u_i = f_i$. In view of (3.2) and Lemma 1.8. □

The solution of (3.1) is unique on $M_\infty$ because the maximum principle holds, see Section 7 of [Ch3].

We now study the Green's functions. Assume $(M_i^n, p_i, \text{Vol}_i) \xrightarrow{d_{GH}} (M_\infty, p, \mu_\infty)$ in the pointed measured Gromov-Hausdorff sense ([Gr], [ChCo2]), where $\text{Ric}_{M_i^n} \geq 0$, $M_i^n$ is complete, noncompact, $n \geq 3$.

**Theorem 3.21.** *Assume, $M_i^n$ also satisfies* $\text{Vol}(B_R(p_i)) > v_0 R^n$. *Then on $M_\infty$ there is a Green's function $G_\infty$, $G_i \to G_\infty$ uniformly on any compact subsets of $M \times M$ that does not intersects with the diagonal.*

*Proof.* Since $n \geq 3$, the Euclidean volume growth condition (0.4) implies that the minimal positive Green's function $G_i$ exists on $M_i^n$ ($\Delta$ is *positive*). Moreover, $G_i$ satisfies the Li-Yau estimate (1.19). So by the Cheng-Yau gradient estimate and the Arzela-Ascoli theorem, for any fixed $x$, for some subsequence (still denoted by $G_i$), we have

$$G_i(x, y) \to G_\infty(x, y), \tag{3.22}$$

uniformly over any compact set in $M \setminus \{x\}$. Clearly, $G_\infty$ satisfies (1.19). We will show $G_\infty$ is in fact well defined, and $G_i \to G_\infty$ as stated in the above theorem.

Assume $f_\infty$ is any Lipschitz function supported in $B_K(p_\infty) \subset M_\infty$, $\mathbf{Lip} f_\infty \leq L$. By Lemma 1.8 and approximation, there is a sequence of $C^2$ functions $\{f_i\}$ with $f_i \to f_\infty$ uniformly, $\mathbf{Lip} f_i \leq 2L$, $\mathrm{supp} f_i \subset B_{2K}(p_i) \subset M_i^n$, $i = 1, 2, ..., \infty$.

Recall on each manifold $M_i^n$ with maximal volume growth condition, the function,

$$u_i(x) = \int_{M_i^n} G_i(x, y) f_i(y) dy, \tag{3.23}$$

solves the Poisson equation

$$\Delta u_i = f_i \ , \ \lim_{x \to \infty} u_i(x) = 0. \tag{3.24}$$

Now by the Li-Yau estimate (1.19) and the Euclidean volume growth condition (0.4), $G_i$ is locally integrable, so $u_i$ is uniformly bounded. The gradient estimate Lemma 1.14 shows that $u_i$ are uniformly Lipschitz:

$$\mathbf{Lip} u_i \leq C(L, K, n). \tag{3.25}$$

So we can find a subsequence of $\{u_i\}$ that converges to some Lipschitz function $u_\infty$ on $M_\infty$. Note that by the Li-Yau estimate, (1.19),

$$|u_i(x)| \leq C'(L, K, n) d(x, p_i)^{2-n}, \quad (i = 1, 2, ..., \infty). \tag{3.26}$$

So by Lemma 3.17, $\Delta u_\infty = f_\infty$ on $M_\infty$. Using the fact Laplacian is linear, by (3.26) and the maximum principle (Section 7 of [Ch3]), it is clear that $u_\infty$ is well defined and $u_i \to u_\infty$ uniformly.

Notice, by (1.19),

$$u_\infty(x) = \int_{M_\infty} G_\infty(x, y) f_\infty(y) dy. \tag{3.27}$$

Since we can choose arbitrary $K$, $f_\infty$, clearly $G_\infty$ is also well defined, $G_i \to G_\infty$ uniformly, off the diagonal. By (3.27) and Lemma 3.17, $G_\infty$ can be interpreted as the minimal positive Green's function on $M_\infty$. □

## 4. Separation of variables on tangent cones

Assume $M_i^n$ is complete noncompact, $\mathrm{Ric}_{M_i^n} \geq 0$ and satisfies (0.4) uniformly, $M_i^n \xrightarrow{d_{GH}} M_\infty$. Recall that by [ChCo1], [ChCo2], every tangent cone of $M_\infty$ is a metric cone. We denote such a cone by $C(X) = \mathbf{R}_+ \times_r X$, here $(X, dx^2)$ is a compact length space with $\mathrm{Diam}\, X \leq \pi$, [ChCo1]. The metric on $C(X)$ is

$$d\rho^2 = dr^2 + r^2 dx^2. \tag{4.1}$$

Here we write $r$ for the distance from the pole $p_\infty = (0, X)$.

The measure $\mu_\infty$ on $C(X)$ is just the $n$-Hausdorff measure, [ChCo2]. Since we can rescale $C(X)$, $\mu_\infty$ induced a natural measure $\mu_X$ on $X$ that obviously satisfies a doubling condition (0.2) (with some different $\kappa$). Moreover, $X$ satisfies the rectifiability properties as stated in Section 5 of [ChCo4].

Also recall from [Ch3], for $f, g \in H_{1,2}$,

$$d(fg) = f \cdot dg + g \cdot df. \tag{4.2}$$

Moreover, from [ChCo4] and [Ch3], if $f$ is a function depending only on $r$ and $g$ is a function independent of $r$, then by the polar identity, one gets $< df, dg >= 0$.

**Lemma 4.3** (Weak Poincare inequality)**.** *For $B_R \subset X$, $3R < \frac{1}{5}$, $f \in H_{1,2}(X)$,*

$$\int_{B_R(x)} |f - f_{x,R}|^2 \le \tau_X R^2 \int_{B_{3R(x)}} |df|^2. \tag{4.4}$$

*Proof.* Define, for $x \in X$,

$$\text{Box}((1,x),a,b) = \{(t,y) \in C(X) | |t-1| < a, d_X(x,y) < b\}. \tag{4.5}$$

Put

$$\text{Box}_1 = \text{Box}((1,x),R,R),\ \text{Box}_2 = \text{Box}((1,x),3R,3R) \subset C(X) \tag{4.6}$$

So $\text{Box}_1 \subset B_{2R}((1,x)) \subset \text{Box}_2$. We extend $f$ to be a $H_{1,2}$ function independent of $r$ on $C(X)$. Assume $f_{\text{Ball}}$ is the average of $f$ on the ball $B_{2R}((1,x)) \subset C(X)$,

$$\begin{aligned} \int_{B_R(x)} |f - f_{x,R}|^2 &= C(n)R^{-1} \int_{\text{Box}_1} |f - f_{x,R}|^2 \le C(n)R^{-1} \int_{\text{Box}_1} |f - f_{\text{Ball}}|^2 \\ &\le C(n)R^{-1} \int_{B_{2R}((1,x))} |f - f_{\text{Ball}}|^2 \le C(n)\tau R \int_{B_{2R}((1,x))} |df|^2 \\ &\le C(n)\tau R \int_{\text{Box}_2} |df|^2 = \tau_X R^2 \int_{B_{3R}(x)} |df|^2. \end{aligned} \tag{4.7}$$

The first and last identity come from the Fubini theorem. Note $f_{x,R}$ is also the average of $f$ over $\text{Box}_1$, and we used the Poincare inequality on $C(X)$ in the middle inequality. ☐

We remark, a weak Poincare inequality is already enough for many purposes. Since $X$ is a length space, by [HaKo] one has (0.3) on $X$. As in [Ch3], [ChCo4], we define a *positive* operator $\Delta_X$ on $X$. Note by (0.2), (0.3) the compact embedding lemma 1.3 is true on $X$. So by the standard elliptic theory, on $X$ we have a basis $\{\phi_j\}_{j=0}^{\infty}$ for $L^2(X)$ and a sequence $\mu_j \to \infty$ such that $\Delta_X \phi_j = \mu_j \phi_j$, compare [Ch3], [ChCo4]. Moreover, one can do Moser iteration on $X$, so $\phi_i$ is Hölder continuous; see [Lin], [GT]. These have applications in Section 6.

Next we show, even the cross section $X$ may not be a manifold, there is still a separation of variables formula for $\Delta$ on $C(X)$. See [Ch1] for the classical case.

Recall that $< \cdot, \cdot >$ is the inner product on $T^*M_\infty$ as in [Ch3], [ChCo4],

**Lemma 4.8.**

$$\Delta(fg) = f\Delta g + g\Delta f - 2 < df, dg > . \tag{4.9}$$

*Proof.* Since $d(fg) = f \cdot dg + g \cdot df$, for any Lipschitz (or $H_{1,2}$) function $\phi$ with compact support, we have (recall $\Delta$ is *positive*)

$$\int < df, g \cdot d\phi + \phi \cdot dg > - \int g\phi\Delta f = 0. \tag{4.10}$$

Exchange the role of $f$ and $g$, we get

$$\int < d(fg), d\phi > - \int \phi(f\Delta g + g\Delta f - 2 < df, dg >) = 0. \tag{4.11}$$

□

Similarly, by $d(f \circ g) = f'(g)dg$, we get

$$\Delta f \circ g = -f''(g)|dg|^2 + f'(g)\Delta g. \tag{4.12}$$

**Lemma 4.13.** *On $C(X)$, $r^{2-n}$ is harmonic away from the pole.*

*Proof.* By the results in Section 4 of [ChCo1], $r^{2-n}$ is the uniform limit of a sequence of harmonic functions $\mathcal{G}$. So by the proof of Lemma 3.17, $r^{2-n}$ is harmonic. □

By the maximum principle on $X$ (Section 7 in [Ch3]), we have

**Lemma 4.14.** *If $X$ is compact, and $\Delta_X f = 0$, then $f$ is a constant.*

**Theorem 4.15.** *Assume $u$ lies in the ring generated by functions of the form $u = fg$ where $f$ depends only on $r$ and $g$ depends only on $x$. Then on $C(X) \setminus \{p_\infty\}$,*

$$\Delta u = -\frac{\partial^2 u}{\partial r^2} - \frac{n-1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\Delta_X u. \tag{4.16}$$

*Proof.* Compare [ChTa1]. By (4.12) and Lemma 4.13, on the cone $C(X)$ we have

$$\Delta f(r) = -f''(r) - f'\frac{n-1}{r}. \tag{4.17}$$

Next we apply Lemma 4.8, recall $< df, dg >= 0$. We pick a test function $\phi$ of the form $\phi = a(r)b(x)$. By scaling we see, $\Delta g(R, x) = R^{-2}\Delta g(1, x)$. Assume $a$ is supported over the interval $[\alpha, \beta]$,

$$\int_{C(X)} < dg, d\phi >= \int_\alpha^\beta (t^{1-n} \int_X t^{-2} < dg, a(t)db > dx)dt, \tag{4.18}$$

here in the second integral we view $g$ and $b$ as functions on the cross section $X = (1, X)$. So we compute

$$\int_{C(X)} < dg, d\phi >= \int_\alpha^\beta (t^{-n-1} \int_X a(t)b(x)\Delta_X g dx)dt. = \int \phi r^{-2}\Delta_X g. \tag{4.19}$$

Since we can choose arbitrary $a, b$, and $\Delta g(R, x) = R^{-2}\Delta g(1, x)$, we get

$$\Delta g(R, x) = R^{-2}\Delta_X g. \tag{4.20}$$

This suffices to complete the proof. □

Using transformation $D_R : (r, x) \mapsto (Rr, x)$, we deduce from the existence and uniqueness of $G_\infty$ that

$$G_\infty(D_R x, D_R y) = R^{2-n} G_\infty(x, y). \tag{4.21}$$

So $G_\infty(p_\infty, x) = d(p_\infty, x)^{2-n} g(x)$ for some Lipschitz function $g$. By (4.16) and Lemmas 4.8, 4.14, $g = C$ is a constant.

**Corollary 4.22.**

$$G_\infty(p_\infty, x) = (n-2)^{-1}\mu_X(X)^{-1} d^{2-n}(p_\infty, x). \tag{4.23}$$

*Proof.* We know $G_\infty(p_\infty, x) = Cd^{2-n}(p_\infty, x)$. We construct a test function $\phi = \phi(r)$ such that $\phi$ is a smooth function of $r$, $\phi = 1$ for $r$ small and $\phi = 0$ for $r \geq 1$. So

$$\begin{aligned} 1 &= \int G(p_\infty, y)\Delta\phi(y) = \int_0^1 (-\phi'' - \frac{n-1}{r}\phi')Cr^{2-n}r^{n-1}\mu_X(X)dr \\ &= -C\mu_X(X)\int_0^1 (n-2)\phi' dr = (n-2)C\mu_X(X). \end{aligned} \tag{4.24}$$

□

**Corollary 4.25.** *Assume* $\Delta u = f$ *on* $B_R(p_\infty) \setminus \{p_\infty\}$, $f \in L^\infty$, *and*

$$\lim_{x\to p} |u(x)| d(x, p_\infty)^{n-2} = 0. \tag{4.26}$$

*Then* $\Delta u = f$ *on* $B_R(p_\infty)$.

*Proof.* By the De Giorgi-Nash-Moser theorem, $u$ is bounded and Hölder continuous. In our case, $G_\infty(p_\infty, x) = Cd(p_\infty, x)^{2-n}$, so the proof goes exactly like the $\mathbf{R}^n$ case (where the maximum principle is used). For details see [Lin]. □

Relation (4.23) implies the Colding-Minicozzi asymptotic formula, [CoMi1], compare [LiTW]. In fact, we rescale the manifold $M^n$ to get a sequence of manifolds that converges to $C(X)$, a tangent cone at infinity, see [ChCo2]. By Theorem 3.21, the new (rescaled) Green's functions converge to the Green's function on $C(X)$.

**Theorem 4.27** (Colding-Minicozzi)**.** *On a noncompact manifold* $M^n$ *with* $\mathrm{Ric}_{M^n} \geq 0$ *and (0.4) we have*

$$\lim_{d(x,p)\to\infty} d(x,p)^{n-2}G(p,x) = (n-2)^{-1}(n \lim_{R\to\infty} R^{-n}\,\mathrm{Vol}(B_R(p)))^{-1}. \tag{4.28}$$

Note the tangent cones may not be unique; in collapsing case, a tangent cone might not be a metric cone, [ChCo2], [Per].

## 5. Heat kernels on noncompact spaces

We assume in this section, all the manifolds $M^n$ are noncompact, satisfying (0.1). On $M^n$, write $H(x, y, t)$ for the heat kernel; we denote by $H_R(x, y, t)$ the Dirichlet heat kernel on the metric ball $B_R(p)$, put $H_R = 0$ outside $B_R(p)$.

One technical issue is, the boundary $\partial B_R(p) = d^{-1}(R)$ may not be smooth, here $d = d(p, \cdot)$. However, we can approximate $d$ by a Morse function $d_\epsilon$, see [Hir], and (assuming) $R$ is not a critical value, etc. So in the sequel we always assume the boundary are smooth.

**Lemma 5.1.** *Assume* $\mathrm{Ric}_{M^n} \geq -(n-1)\Lambda$. *Then there is a function* $\epsilon(t, \Lambda, R)$ *with* $\lim_{R\to\infty} \epsilon(t, \Lambda, R) = 0$ *for* $t > 0$, *and*

$$\int_{M - B_R(x)} H(x, y, t)dy \leq \epsilon(t, \Lambda, R). \tag{5.2}$$

*Proof.* By the Bishop-Gromov inequality, it's easy to see

$$\operatorname{Vol}(B_{\sqrt{t}}(x)) \leq C_1(n,\Lambda,t)e^{C_2(n,\Lambda,t)d(x,y)}\operatorname{Vol}(B_{\sqrt{t}}(y)). \tag{5.3}$$

Put $s_\Lambda(r) = (1/\sqrt{\Lambda})\sinh\sqrt{\Lambda}r$. We now use the Li-Yau estimate (1.16):

$$\begin{aligned}&\int_{M-B_R(x)} H(x,y,t)dy\\ &\quad\leq C'(n,\Lambda,t)\int_{M-B_R(x)}\operatorname{Vol}^{-1}(B_{\sqrt{t}}(x))e^{-d(x,y)^2/5t}e^{C_2(n,\Lambda,t)d(x,y)}\\ &\quad= C'(n,\Lambda,t)\int_R^\infty e^{-r^2/5t}e^{C_2r}A(r)s_\Lambda^{n-1}(r)dr\Big/\int_0^{\sqrt{t}}A(r)s_\Lambda^{n-1}(r)dr\\ &\quad\leq C'\int_R^\infty e^{-r^2/5t}e^{C_2r}s_\Lambda^{n-1}(r)dr\Big/\int_0^{\sqrt{t}}s_\Lambda^{n-1}(r)dr=\epsilon(t,\Lambda,R).\end{aligned} \tag{5.4}$$

Here $A(r)s_\Lambda^{n-1}(r)$ is the surface area element of $\partial B_R(x)$. We used the fact $A(r)$ is non-increasing (Bishop-Gromov inequality) and assumed, without lose of generality, $R > \sqrt{t}$. □

**Lemma 5.5.** *Let $(M^n, p)$ be a noncompact complete manifold. Then*

$$\lim_{R\to\infty} H_R(x,\cdot,t) = H(x,\cdot,t). \tag{5.6}$$

*The convergence is uniform, and uniformly in $L^1$, on any finite interval $t \in [0,T]$.*

*Proof.* Assume $R > \max\{T, 2d(x,p)\}$. Put

$$M(R) = \sup\{H(x,y,t)|y \in \partial B_R(x), 0 < t \leq T\}, \tag{5.7}$$

by (1.17) and volume comparison we have

$$M(R) \leq \sup_{0<t\leq T} C(n,\Lambda,T)t^{-C_1(n)}e^{-R^2/5t}e^{C_2(n,\Lambda,T)R}\operatorname{Vol}(B_R(p))^{-1}, \tag{5.8}$$

so $\lim_{R\to\infty} M(R)\operatorname{Vol}(B_R(p)) = 0$. By the maximum principle,

$$H(x,y,t) - M(R) \leq H_R(x,y,t) \leq H(x,y,t). \tag{5.9}$$

Combining this with Lemma 5.1, we have

$$\|H_R(x,\cdot,t) - H(x,\cdot,t)\|_{L^1} < \epsilon(n,\Lambda,T,R), \tag{5.10}$$

and $\lim_{R\to\infty}\epsilon(n,\Lambda,T,R) = 0$. □

Assume $\lambda_j$ is the $j$-th Dirichlet eigenvalue of the Laplacian on $B_R(p)$, $\phi_j$ is the corresponding eigenfunction, $\|\phi_j\|_{L^2(B_R(p))} = 1$.

**Lemma 5.11.** *There exists a constant $C(n,\Lambda,R)$ such that*

$$C(n,\Lambda,R)^{-1}R^{-2}k^{\frac{2}{n}} \leq \lambda_k \leq C(n,\Lambda,R)R^{-2}k^2. \tag{5.12}$$

*Proof.* Since $R$ fixed, we have $\operatorname{Vol}(B_r(x)) \geq C_0(n,\Lambda,R)r^n\operatorname{Vol}(B_R(p))$, for $r < 2R$ and $B_r(x)$ with nonempty intersection with $B_R(p)$. Then since $H_R \leq H$, we can follow the heat kernel argument as in page 178 of [SY] to get the lower bound of $\lambda_k$.

The upper bound follows from an argument of Cheng, see page 105 of [SY]. □

**Lemma 5.13.** *For any $N > 0$, there is a function $\epsilon(N, \Lambda, R, \delta)$ such that for any fixed $R$, $\lim_{\delta\to 0} \epsilon(N, \Lambda, R, \delta) = 0$, And for $k$ such that $\lambda_k < N$,*

$$\int_{A(p,R-\delta,R)} |\phi_k|^2 \le \epsilon(N, \Lambda, R, \delta). \tag{5.14}$$

*Here $A(p, R-\delta, R)$ is the annulus $\{z|R-\delta \le d(p,z) \le R\}$.*

*Proof.* By (1.16) and the Bishop-Gromov inequality, when $t = 1$,

$$\begin{aligned} \int_{\mathrm{A}(p,R-\delta,R)} |\phi_k|^2 &\le e^{\lambda_k} \int_{\mathrm{A}(p,R-\delta,R)} H(x,x,1)dx \\ &\le e^N \int_{\mathrm{A}(p,R-\delta,R)} \frac{C(n,\Lambda,R)}{\mathrm{Vol}(B_R(p))} dx \le \epsilon(N,\Lambda,R,\delta). \end{aligned} \tag{5.15}$$

□

As before, assume $M_i^n \xrightarrow{d_{GH}} M_\infty$ in the pointed measured Gromov-Hausdorff sense, $M_i$ is noncompact, satisfies (0.1). Write $\lambda_{j,i}$ for the $j$-th Dirichlet eigenvalue over $B_R(p_i) \subset M_i^n$. $\phi_{j,i}$ is the corresponding eigenvalue:

$$\Delta\phi_{j,i} = \lambda_{j,i}\phi_{j,i}; \quad \int_{B_R(p_i)} \phi_{j,i}\phi_{k,i} = \delta_{jk}. \tag{5.16}$$

**Lemma 5.17.** *For fixed $j, k > 0$, assume (for a subsequence of the eigenvalues), $\lambda_{j,i} \to \lambda_{j,\infty}$, $\lambda_{k,i} \to \lambda_{k,\infty}$. Then there is a subsequence (denoted also by $\phi_{j,i}, \phi_{k,i}$) that converges uniformly on compact subsets of $B_R$, and also in $L^2(B_R)$, to two locally Lipschitz functions $\phi_{j,\infty}, \phi_{k,\infty}$. Moreover,*

$$\Delta\phi_{j,\infty} = \lambda_{j,\infty}\phi_{j,\infty},\ \Delta\phi_{k,\infty} = \lambda_{k,\infty}\phi_{k,\infty},\ \int_{B_R(p)} \phi_{j,\infty}\phi_{k,\infty} = \delta_{jk}. \tag{5.18}$$

*Proof.* The results is clear in view of Lemma 5.11, Lemma 1.14 and Lemma 3.17. The $L^2$ convergence and the orthonormal property for the limit functions are implied by locally uniform convergence and Lemma 5.13. □

By Lemma 5.11, we can assume, after passing to a subsequence, that *every* eigenvalue and eigenfunction converge:

$$\lim_{i\to\infty} \lambda_{j,i} = \lambda_{j,\infty},\ \lim_{i\to\infty} \phi_{j,i} = \phi_{j,\infty}. \tag{5.19}$$

Write

$$H_{R,\infty} = \sum_{j=1}^{\infty} e^{-\lambda_{j,\infty}t}\phi_{j,\infty}(x)\phi_{j,\infty}(y). \tag{5.20}$$

For all fixed $t, x$, by Lemma 5.11 and Lemma 1.11, Lemma 1.14,

$$H_{R,i}(x,\cdot,t) \to H_{R,\infty}(x,\cdot,t). \tag{5.21}$$

The convergence is in $L^2$, and is locally uniform. Note we don't know if $H_{R,\infty}$ (and $\phi_{j,\infty}$, $\lambda_{j,\infty}$) is well defined. For the moment (before Lemma 5.40), we fix, by a diagonal argument, *one* sequence $R_k \to \infty$, and *one* subsequence $\{M_{i_v}^n\}$ of $\{M_i^n\}$ such that for each $k$, $H_{R_k,i} \to H_{R_k,\infty}$. For simplicity, we just write $\{M_i^n\}$ for this subsequence of manifolds. So by the results on smooth manifolds, for $R_j < R_k$,

$$0 \le H_{R_j,\infty}(x,y,t) \le H_{R_k,\infty}(x,y,t) \le \frac{C(n,\Lambda)e^{-d^2(x,y)/5t}e^{C\Lambda t}}{\mathrm{Vol}_\infty^{1/2}(B_{\sqrt{t}}(x))\,\mathrm{Vol}_\infty^{1/2}(B_{\sqrt{t}}(y))}. \tag{5.22}$$

Thus we can also assume that the nondecreasing sequence $H_{R_j,\infty}$ converges pointwise to some function $H_\infty$. We will prove that $H_\infty$ is well defined.

By (5.9) and the locally uniform convergence of $H_{R,i}$ to $H_{R,\infty}$ (5.21), the Li-Yau estimate (1.16) is also true for $H_\infty$:

$$0 \le H_\infty(x,y,t) \le \frac{C(n,\Lambda)e^{-d^2(x,y)/5t}e^{C\Lambda t}}{\mathrm{Vol}_\infty^{1/2}(B_{\sqrt{t}}(x))\,\mathrm{Vol}_\infty^{1/2}(B_{\sqrt{t}}(y))}. \tag{5.23}$$

Note we need to renormalize the measures whenever $\{M_i^n\}$ is collapsing. Clearly, when $\mathrm{Ric}_{M_i^n} \ge 0$, we also have a lower bound of $H_\infty$ as in (1.18).

**Corollary 5.24.**

$$\int_{M_\infty} H_\infty(x,z,s)H_\infty(z,y,t-s)dz = H_\infty(x,y,t). \tag{5.25}$$

*Proof.* By (5.21), (5.25) is true for $H_{R,\infty}$. Write $H_\infty(x,z,s) = H_{R,\infty}(x,z,s) + \epsilon_R^1(z)$, similarly $H_\infty(z,y,t-s) = H_{R,\infty}(z,y,t-s) + \epsilon_R^2(z)$, here $H_{R,\infty} = 0$ outside $B_R(p_\infty)$, $\epsilon_R^1, \epsilon_R^2 \ge 0$ are two functions. In view of Lemmas 5.1, 5.5, (5.21) and (5.23),

$$\limsup_{R\to\infty}(\|\epsilon_R^1(z)\|_{L^1} + \|\epsilon_R^2(z)\|_{L^1}) = 0, \quad \|\epsilon_R^1(z)\|_{L^\infty} + \|\epsilon_R^2(z)\|_{L^\infty} < C(t,s,M_\infty). \tag{5.26}$$

Now (5.25) is clear. □

**Corollary 5.27.**

$$\int_{M_\infty} H_\infty(x,y,t)dy = 1. \tag{5.28}$$

*Proof.* By (5.21), Lemmas 5.1 and 5.5. □

**Lemma 5.29.** *For any Lipschitz function $f$ with compact support,*

$$|\int_{M_\infty} H_{R,\infty}(x,y,t)f(y)dy - f(x)| \le \epsilon(t, \|f\|_{L^\infty}, \mathbf{Lip}f). \tag{5.30}$$

*Here for any $F, L > 0$, $\lim_{t\to 0}\epsilon(t,F,L) = 0$. The conclusion is also true for $H_\infty$.*

*Proof.* By an argument similar to those given in Lemma 5.1 and Lemma 5.5. Note on smooth manifolds, when $t \to 0$, the integral of $H_R$ is smaller than, but almost equal to 1, and tends to concentrate on smaller and smaller balls centered at $x$. In view of (5.21) and the Li-Yau estimate (5.23), we easily get (5.30). □

Let the Sobolev space $\overset{\circ}{H}_{1,2}(B_R(p_\infty))$ be defined as in [Ch3], i.e. the $H_{1,2}$ closure of the set of Lipschitz functions supported in the interior of $B_R(p_\infty)$,

**Lemma 5.31.** *The space $\overset{\circ}{H}_{1,2}(B_R(p_\infty))$ is contained in $\Phi$, the $L^2$-linear span of functions $\phi_{j,\infty}$. In particular, any Lipschitz function with support in $B_{R-\delta}$ lies in $\Phi$.*

*Proof.* If not, by approximation, we have a Lipschitz function $f_\infty$ with compact support and an $\epsilon > 0$ such that

$$\sum_{j=1}^{\infty}(\int_{B_R(p_\infty)} f_\infty \phi_{j,\infty})^2 < (1-3\epsilon)\|f_\infty\|_{L^2}^2. \tag{5.32}$$

Using Lemma 1.8, we can transplant $f_\infty$ back to a Lipschitz function, $f_i$, on $M_i^n$, with compact support which is close to $f_\infty$ in $L^\infty$, such that the energy of $f_i$ is close to that of $f_\infty$. Write

$$f_i = \sum_{j=1}^{N} a_{j,i}\phi_{j,i} + R_{N,i},\ \ R_{N,i} = \sum_{j=N+1}^{\infty} a_{j,i}\phi_{j,i}. \tag{5.33}$$

Notice,

$$\lim_{i\to\infty} a_{j,i} = \int_{M_\infty} f_\infty \phi_{j,\infty}. \tag{5.34}$$

So by the min-max principle and Lemma 5.11, $\lim_{i\to\infty}\|\nabla f_i\|_{L^2} = \infty$, we get a contradiction to the construction of $f_i$, Lemma 1.8. □

Remark, it is not clear if we have $\phi_{j,\infty} \in \overset{\circ}{H}_{1,2}$.

Now for Lipschitz functions $f_i$ with compact support in $B_R(p_i) \subset M_i^n$ $(i = 1, 2, ..., \infty)$, $f_i \to f_\infty$ uniformly, we have

$$f_i = \sum_{j=1}^{\infty} a_{j,i}\phi_{j,i}, \qquad a_{j,i} = \int_{M_i^n} f_i \phi_{j,i}. \tag{5.35}$$

So $a_{j,i} \to a_{j,\infty}$. Clearly,

$$\int_{B_R(p_i)} H_{R,i}(x,y,t) f_i(y)dy = \sum_{j=1}^{\infty} e^{-\lambda_{j,i}t} a_{j,i}\phi_{j,i}(x). \tag{5.36}$$

We say $h(x,t)$ is a *locally strong* solution, if $h$ continuous, Lipschitz in $x$, $\frac{\partial h}{\partial t}$ exists, continuous on $M \times \mathbf{R}^+$, and when $t$ fixed, $-\Delta h = \frac{\partial h}{\partial t}$, i.e.

$$\int_\Omega \psi\frac{\partial h}{\partial t} + \int_\Omega < d_x h, d_x\psi >= 0, \tag{5.37}$$

for all Lipschitz functions $\psi$ with compact support.

By Lemma 5.11, Lemma 1.11 and 1.14,

$$\lim_{k\to\infty}\sum_{j=k} |e^{-\lambda_j t}\phi_j(x) d_y\phi_j(y)| = 0. \tag{5.38}$$

So $H_{R,i}$ is a locally strong solution of the heat equation. Similarly the function,

$$h_i(x,t) = \int_{B_R} H_{R,i}(x,y,t) f_i(y)dy \qquad (i = 1, 2, ..., \infty), \tag{5.39}$$

is also a locally strong solution. Note for the case $i = \infty$ we used also Lemma 5.17.

For locally strong solutions on $M_\infty$, there is also a weak maximum principle:

**Lemma 5.40.** *Assume $h$ is a locally strong solution on $B_{2R}\times[0,T+1]$, then if*

$$h|_{B_R\times\{0\}}\le 0,\ \ h|_{\partial B_R\times[0,T]}\le 0. \tag{5.41}$$

*Then $h\le 0$ on $B_R\times[0,T]$.*

*Proof.* Define

$$m(s)=\sup\{h(x,s)|x\in B_R,\frac{\partial h}{\partial t}(x,s)\le 0\}. \tag{5.42}$$

Since $h,\frac{\partial h}{\partial t}(x,s)$ are continuous functions, it's easy to show that $m$ is nonincreasing and $m(0)=0$ implies $m(s)\le 0$ for all $s>0$. Now by the weak maximum principle for Poisson equations (see [GT], [Ch3] or (5.37)), we have, when $s$ fixed,

$$\sup\{h(x,s)|x\in B_R\}=m(s)\le 0. \tag{5.43}$$

□

Now we can address the uniqueness of $H_\infty$. Recall that $(M_i,p_i)\to(M^\infty,p_\infty)$. Assume for $R>0$, we got two limits $H^{(1)}_{4R,\infty}$, $H^{(2)}_{4R,\infty}$ through different subsequences of manifolds.

**Theorem 5.44.** *For $x,y\in B_R(p_\infty)$, $t<T$, there is an $\epsilon(R)>0$ such that*

$$\lim_{R\to\infty}\epsilon(R)=0, \tag{5.45}$$

$$H^{(1)}_{4R,\infty}(x,y,t)<H^{(2)}_{4R,\infty}(x,y,t)+\epsilon(R). \tag{5.46}$$

*Proof.* We can assume $R>T^2>t^2$ and $R>4$. Assume (5.46) is not true, then there is a point $a\in B_R(p_\infty)$ and $0<r<1$ such that

$$H^{(1)}_{4R,\infty}(x,y,t)\ge H^{(2)}_{4R,\infty}(x,y,t)+\epsilon(R), \tag{5.47}$$

for $y\in B_{2r}(a)$. We then construct a test function $f\ge 0$ such that, $f$ Lipschitz, supported in $B_r(a)$,

$$2\int_{B_r(a)}f\ge \mathrm{Vol}(B_r(a))\sup_{B_r(a)}f. \tag{5.48}$$

Clearly, for $R<\infty$, the functions,

$$F_k(z,s)=\int_{B_r(a)}H^{(k)}_{4R,\infty}(z,y,s)f(y)dy,\ (k=1,2), \tag{5.49}$$

are locally strong solutions of the heat equation, and (by the construction of $f$),

$$F_1(x,t)\ge F_2(x,t)+\epsilon\int_{B_r(a)}f\ge F_2(x,t)+\frac{\epsilon(R)}{2}\mathrm{Vol}(B_r(a))\sup_{B_r(a)}f. \tag{5.50}$$

For a point $z$ near $\partial B_{2R}(p_\infty)$, say $d(z,p)=2R$, $d(a,z)\ge R$,

$$F_k(z,s)<\mathrm{Vol}(B_r(a))\frac{C(n)}{\mathrm{Vol}(B_{\sqrt{s}}(z))}e^{-R^2/5s}e^{CR}\sup_{B_r(a)}f,\ (k=1,2). \tag{5.51}$$

By a standard argument of the Bishop-Gromov inequality,

$$F_k(z,s)<C_1(n)\frac{\mathrm{Vol}(B_r(a))}{\mathrm{Vol}(B_1(p))}s^{-\frac{n}{2}}e^{-R^2/5s}e^{C'R}\sup_{B_r(a)}f,\ (k=1,2). \tag{5.52}$$

Next we consider the case that $s$ is small. Since $f$ is fixed, by (5.30), $F_k \to f$ uniformly on $B_{2R}(p)$ when $s \to 0$.

In view of the weak maximum principle on $B_{2R}(p_\infty) \times [0, T]$ (Lemma 5.40), clearly we should choose $\epsilon(R)$ such that that for $0 \le s \le T$,

$$\frac{C_1(n)}{\mathrm{Vol}(B_1(p))} s^{-\frac{n}{2}} e^{-R^2/5s} e^{C'R} < \frac{\epsilon(R)}{4}, \tag{5.53}$$

by the maximum principle we got a contradiction to (5.50). □

**Theorem 5.54.** *$H_\infty$ is well defined. For fixed $t > 0$, $x_i \to x_\infty$, we have $H_i(x_i, \cdot, t) \to H_\infty(x_\infty, \cdot, t)$ in $L^1$. When $H_\infty$ is continuous, this convergence is also uniform.*

*Proof.* By the previous theorem and the construction of $H_\infty$ (compare (5.9)), we see $H_\infty$ is independent of the choice of subsequences, so well defined.

We already know, by (5.9), (5.21), (5.23), that locally $H_i \to H_\infty$ in $L^1$. The proof of global $L^1$ convergence is similar with Lemma 5.1, Lemma 5.5, using (1.18), (5.23).

Recall (see [SY] Chapter 4), there is a Harnack inequality

$$H_i(x, y_1, t_1) \le H_i(x, y_2, t_2)(\frac{t_2}{t_1})^n \exp(\frac{d^2(y_1, y_2)}{4(t_2 - t_1)} + C(n, \Lambda)(t_2 - t_1)), \tag{5.55}$$

for $0 < t_1 < t_2$. If $H_\infty$ is continuous, then locally $H_\infty$ is uniformly continuous (especially, with respect to $t$), clearly by (5.55) the convergence $H_i \to H_\infty$ must be uniform, compare with (5.23). □

We now want to interpret the meaning of $H_\infty$. Recall from [ChCo4] and [Ch3], $\Delta$ is a *positive self-adjoint operator*. So $-\Delta$ generates a semigroup $e^{-t\Delta}$.

Assume $f_i$ is supported in $B_K(p_i) \subset B_R(p_i)$. Use the notation in (5.35), define

$$W_{R,i}(t) f_i(x) = \sum_{j=1}^{\infty} a_{j,i} \cos(\sqrt{\lambda_{j,i}} t) \phi_{j,i}. \tag{5.56}$$

By the finite speed of propagation (see [Ta]), when $t$ is fixed and $R > K + t$, $W_{R,i}(t)f$ is independent of $R$. We write $W_i(t)f$ for $W_{R,i}(t)f$ with $R$ big. For $i < \infty$,

$$e^{-t\Delta} f_i(x) = \int_{M_i^n} H_i(x, y, t) f_i(y) dy = \int_{-\infty}^{\infty} e^{-s^2/4t} W_i(s) f_i(x) ds, \tag{5.57}$$

see [CGT], [Ta]. Define

$$W_{R,\infty}(t) f_\infty(x) = \sum_{j=1}^{\infty} a_{j,\infty} \cos(\sqrt{\lambda_{j,\infty}} t) \phi_{j,\infty}. \tag{5.58}$$

We notice that $W_{R,i}$ $(i = 1, 2, ..., \infty)$ does not increase $L^2$ norm, and we should use Lemma 1.8 and approximation to construct $C^2$ functions $f_i$ on $M_i^n$ that converges to $f_\infty$. Clearly, $W_{R,i} f_i \to W_{R,\infty} f_\infty$ in $L^2$. We remark that generally, we don't know if $W_{R,\infty}$ is well defined.

**Theorem 5.59.** *If the limit $M_\infty$ is a smooth manifold, and the limit measure is the canonical measure on $M_\infty$, then $H_\infty$ is the heat kernel on $M_\infty$.*

*Proof.* In the noncollapsing case, by Colding's theorem [Co], the limit measure is the canonical measure on $M_\infty$; when $M_\infty = \mathbf{R}^k$ for some $k$, the limit measure is also a multiple of the standard Lebesgue measure on $\mathbf{R}^k$, see [ChCo2]. In these cases, the Laplacian we defined on $M_\infty$ is the same one from the original smooth structure of $M_\infty$.

Pick any $C_0^\infty$ function $f$ supported in $B_R$, So

$$\int_{M_\infty} (\Delta^k) f \phi_{j,\infty} = (\lambda_{j,\infty})^k \int_{M_\infty} f \phi_{j,\infty} = (\lambda_{j,\infty})^k a_{j,\infty}. \tag{5.60}$$

Since $(\Delta^k) f \in C_0^\infty$, we have for all $k$, $\lim_{j\to\infty} (\lambda_{j,\infty})^k a_{j,\infty} = 0$. By Lemma 5.11, we have for all $k$, $\lim_{j\to\infty} j^k a_{j,\infty} = 0$. So $W_{R,\infty}(t)f$ is a classical solution of the wave equation, when $R$ is big enough, $W_{R,\infty}(t)f = W_\infty(t)f$ is independent of $R$. Since $M_\infty$ is a smooth manifold,

$$e^{-t\Delta} f(x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-s^2/4t} W_\infty(s) f(x) ds. \tag{5.61}$$

In view of (5.57), combined with the fact $W_\infty$ does not increase $L^2$ norm and $H_i(x,y,t)$ converges uniformly to $H_\infty(x,y,t)$, we have

$$e^{-t\Delta} f(x) = \int_{M_\infty} H_\infty(x,y,t) f(y) dy. \tag{5.62}$$

That concludes the proof. □

## 6. Laplacian on metric cones

In this section, we assume $M_i^n \xrightarrow{d_{GH}} C(X)$ where $C(X)$ is a metric cone; $\mathrm{Ric}_{M_i^n} \geq 0$, $M_i^n$ is complete noncompact and satisfies (0.4) uniformly, $n \geq 3$.

Write $p_\infty$ for the pole of $C(X)$, define $r(x) = d(x, p_\infty)$.

**Theorem 6.1.** *If $M_\infty = C(X)$, then $H_\infty$ is the integral kernel of the semigroup $e^{-t\Delta}$.*

*Proof.* In view of (5.23), (5.25) and the Young's inequality, one can define a semigroup $E(t)$ on $L^2(M_\infty)$ by

$$E(t) f(x) = \int_{M_\infty} H_\infty(x,y,t) f(y) dy. \tag{6.2}$$

We want to compare $E(t)$ with $e^{-t\Delta}$. First, by Theorem 3.21, (1.18) and (5.23), one easily get

$$G_\infty(x,y) = \int_0^\infty H_\infty(x,y,t) dt. \tag{6.3}$$

Pick any $L^2$ function $f$ with compact support. Write

$$F(x) = \int_{M_\infty} G_\infty(x,y) f(y) dy. \tag{6.4}$$

We compute

$$\begin{aligned}\frac{E(t)F-F}{t} &= \int_{M_\infty}\left(\frac{H_\infty(x,y,t)}{t}\right)\int_{M_\infty}\int_0^\infty H_\infty(y,z,s)f(z)dsdzdy \\ &\quad -\frac{1}{t}\int_{M_\infty}\int_0^\infty H_\infty(x,z,s)f(z)dsdz \\ &= -\frac{1}{t}\int_0^t\int_{M_\infty} H_\infty(x,z,s)f(z)dzds.\end{aligned} \tag{6.5}$$

So by (0.4), (5.23), (5.30) and the Young's inequality we have

$$\lim_{t\to 0}\frac{E(t)F-F}{t}\to -f. \tag{6.6}$$

in $L^2$ and $L^1$.

Now we use the assumption that $M_\infty = C(X)$ is a noncollapsed cone. Recall the results in Section 4, we can construction a function $\phi=\phi(r)$ such that $\phi$ is a smooth function of $r$, where $r(x)=d(p_\infty,x)$ is the distance from the pole, and

$$\phi(r)=1 \text{ if } r<R,\ \phi(r)=0 \text{ if } r\geq R+2,\ \nabla\phi\leq C_0\sqrt{\phi}. \tag{6.7}$$

So on $M_\infty=C(X)$ we have $\Delta\phi=-\phi''-(n-1)\phi'/r$. This function can serve as a cut off function.

We prove, if $F, f=\Delta F\in L^2$ have compact support, then

$$F=\int_{C(X)}G_\infty(x,y)f(y)dy. \tag{6.8}$$

In fact, assume $\{f_k\}$ is a sequence of Lipschitz functions, $f_k\to f$ in $L^2$, and all $f_k$ together with $f,F$ are supported in the ball $B_K(p_\infty)$. So the function

$$F_k=\int_{C(X)}G_\infty(x,y)f_k(y)dy, \tag{6.9}$$

satisfies $\Delta F_k=f_k$ by the discussion in Section 3. Consider the equation $\Delta(F_k-F)=f_k-f$, i.e.

$$\int_{C(X)}<dF_k-dF,du>-\int_{C(X)}(f_k-f)u=0, \tag{6.10}$$

for any $u\in\overset{\circ}{H}_{1,2}$. We set $u=\phi(F_k-F)$, so $du=d\phi(F_k-F)+\phi(dF_k-dF)$. By the Schwartz inequality,

$$\begin{aligned}\|\sqrt{\phi}d(F_k-F)\|_{L^2}^2 - C_0\|(F_k-F)|_{\mathrm{A}(R,R+2)}\|_{L^2}\|\sqrt{\phi}d(F_k-F)|_{\mathrm{A}(R,R+2)}\|_{L^2} \\ -\|(f_k-f)|_{B_K}\|_{L^2}\|(F_k-F)|_{B_K}\|_{L^2}\leq 0,\end{aligned} \tag{6.11}$$

here $\mathrm{A}(R,R+2)$ is the annulus $\{x|R\leq r(x)\leq R+2\}$. Note we have a definite bound for $\|F_k|_{B_K}\|_{L^2}$ by (1.19) and the Young's inequality. Note also by (1.19) we get, for $R>K$,

$$\begin{aligned}\|(F_k-F)|_{\mathrm{A}(R,R+2)}\|_{L^2} &= \|F_k|_{\mathrm{A}(R,R+2)}\|_{L^2} < C(n,\|f\|_{L^1})(R^{4-2n}R^{n-1})^{1/2} \\ &= C(n,\|f\|_{L^1})R^{(3-n)/2} < C(n,\|f\|_{L^1}),\end{aligned} \tag{6.12}$$

since $n \geq 3$. So first, we get that $\|d(F_k - F)\|_{L^2} < \infty$ by letting $R \to \infty$. Then by letting $k \to \infty$, we have $\|d(F_k - F)\|_{L^2} \to 0$, since we can choose $R$ in (6.11) such that $\|\sqrt{\phi}d(F_k - F)|_{A(R,R+2)}\|_{L^2}$ small.

Now by the $(2,2)$-Poincare inequality, (0.4), (1.19) and Young's inequality, $F_k \to F$ in $L^2$ on compact sets. Also notice, on any compact sets, the right hand side of (6.9) converges to the right hand side of (6.8) in $L^2$, by the Young inequality (however, in view of (1.19), these convergences might not be globally $L^2$). That's enough to imply (6.8).

Next we compute, for $f = \Delta F$, $F, f \in L^2$,

$$\begin{aligned}\|\Delta(\phi F) - f\|_{L^2} &\leq \|F\Delta\phi\|_{L^2} + \|(\phi - 1)f\|_{L^2} + 2\| < d\phi, dF > \|_{L^2} \\ &\leq C(n)\|F|_{A(R,R+2)}\|_{L^2} + \|f|_{A(R,R+2)}\|_{L^2} + C_0\|dF|_{A(R,R+2)}\|_{L^2}.\end{aligned} \tag{6.13}$$

Similar to (6.11), one shows $\|dF\|_{L^2} < \infty$. So if $R \to \infty$, we have $\phi F \to F$ and $\Delta(\phi F) \to f = \Delta F$ in $L^2$. Moreover, by (6.8),

$$\phi F(x) = \int_{C(X)} G_\infty(x, y)\Delta(\phi F)(y)dy. \tag{6.14}$$

So the computation (6.5) is valid for the functions $\phi F$ and $\Delta(\phi F)$:

$$\lim_{t\to 0} \frac{E(t)\phi F - \phi F}{t} = -\Delta(\phi F) \tag{6.15}$$

in $L^2$. We already know $E(t)$ in (6.2) is a semigroup, its infinitesimal generator is a *closed* operator (see [Ta]). So by the above computations, this infinitesimal generator must be the self-adjoint operator $-\Delta$ on $C(X)$. □

By the discussion in the beginning of Section 4, we have an eigenfunction expansion of Laplacian on the unit cross section $X$. We denote by $\phi_j$ $(j = 0, 1, 2, ...)$ the renormalized eigenfunctions with eigenvalues $\mu_j > 0$, note $\phi_0 = \mathrm{Vol}(X)^{-1/2}$. $\mu_j \to \infty$ when $j \to \infty$.

Put $d = \mathrm{Diam}\, X$. Using an argument of Gromov (see [Gr], and Theorem 4.8 of [Ch3]), we have a more precise estimate of $\mu_j$:

$$\mu_j > C(\tau, \kappa)^{-1} d^{-2} j^{\frac{2}{\kappa}}. \tag{6.16}$$

On the other hand, on each ball $B_r(x_k)$ of radius $r = d/2(j + 2)$ on $X$, we define a Lipschitz function $\psi_k$ supported in $B_r(x_k)$ using MacShane's lemma ([Ch3], [ChCo3]):

$$\psi_k(x_k) = r,\ \psi_k(\partial B_r(x_k)) = 0,\ \mathbf{Lip}\psi_k = 1, \tag{6.17}$$

so we can follow the argument of Cheng (see p.105 of [SY]), and get

$$\mu_j \leq C(\kappa) j^2 d^{-2}. \tag{6.18}$$

Now we can use Moser iteration, $|\phi_j|$ is bounded by a definite power of $j$:

$$|\phi_j| \leq C(d, \kappa, \tau) j^{N(\tau,\kappa)}. \tag{6.19}$$

Moreover, $\phi_j$ is Hölder continuous, see [GT], [Lin].

Write $\nu_j = \sqrt{\mu_j + \alpha^2}$, here $m = n - 1$, $\alpha = (1 - m)/2$. We write $x, y$ in polar coordinates, $x = (r_1, x_1), y = (r_2, x_2)$.

**Theorem 6.20.**

$$H_\infty = (r_1 r_2)^\alpha \sum_{j=0}^{\infty} (\frac{1}{2t}) e^{-(r_1^2+r_2^2)/4t} I_{\nu_j}(\frac{r_1 r_2}{2t}) \phi_j(x_1) \otimes \phi_j(x_2). \tag{6.21}$$

*Here $I_{\nu_j}$ are the modified Bessel functions:*

$$I_\nu(z) = (\frac{z}{2})^\nu \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(\nu+k+1)} (\frac{z}{2})^{2k}. \tag{6.22}$$

In our case $\Delta$ is a self-adjoint operator on the whole cone $C(X)$, namely, including the pole $p_\infty$. By Corollary 4.25, the separation of variable formula (4.16) works for $u = f(r)g(x)$ on the whole $C(X)$ if $u$ and $\Delta u$ are bounded on $C(X) \setminus \{p_\infty\}$. So the heat kernel on $M_\infty$ has the expression as on the right hand side of (6.21); the proof goes exactly like the classical case, see [Ch1], [Ch2] page 592, [ChTa1] and [Ta] chapter 8, we omit the details. By Theorem 6.1, we have (6.21).

By Stirling's formula, (6.16) and (6.18), we see the series (6.21) converges uniformly, when $t$ is bounded away from 0 and $r_1, r_2$ stay bounded. In particular, $H_\infty$ is continuous, so by Theorem 5.54 we have $H_i \to H_\infty$ uniformly.

If one of the two points $x$ and $y$, say, $y$, is the pole $p_\infty$, then there is only one term in (6.21). Note $\nu_0 = -\alpha = (m-1)/2$, $m = n-1$,

$$H_\infty(p_\infty, x, t) = (\frac{1}{4\pi t})^{\frac{n}{2}} e^{-(r^2)/4t} \frac{2\pi^{n/2}}{\Gamma(n/2)} (\mathrm{Vol}(X))^{-1}. \tag{6.23}$$

As a corollary, we get a new proof of Li's asymptotic formula for heat kernels [Li1]:

**Corollary 6.24** (Li)**.** *Assume $M^n$ is a complete noncompact manifold satisfying (0.4),* $\mathrm{Ric}_{M^n} \geq 0$*. Then*

$$\lim_{t\to\infty} \mathrm{Vol}(B_{\sqrt{t}}(p)) H(p, y, t) = (4\pi)^{-n/2} \omega_n. \tag{6.25}$$

*$\omega_n$ is the volume of the unit ball in $\mathbf{R}^n$.*

*Proof.* Notice,

$$\lim_{t\to\infty} \mathrm{Vol}(B_{\sqrt{t}}(p)) t^{-n/2} = v_0 = n^{-1} \mathrm{Vol}(X). \tag{6.26}$$

So we need to show,

$$\lim_{t\to\infty} t^{n/2} \mathrm{Vol}(X) H(p, y, t) = (4\pi)^{-n/2} n \omega_n. \tag{6.27}$$

Assume $t_i \to \infty$, $M_i^n = (M^n, p, t_i^{-1} dx^2) \xrightarrow{d_{GH}} C(X)$ for some metric cone $C(X)$; see [ChCo1]. The heat kernel $H_i(p, x, t)$ on $M_i^n$ is

$$H_i(p, y, 1) = t^{n/2} H(p, y, t). \tag{6.28}$$

Here we identify $p, x \in M_i^n$ with $p, x \in M$, however, $d_{M_i^n}(p, x) = t_i^{-1/2} d_M(p, x)$, $d_{M_i^n}$ is the distance on $M_i^n$. In particular, $d_{M_i^n}(p, x) \to 0$ as $i \to \infty$. Since $M_i^n \xrightarrow{d_{GH}} C(X)$, by Theorem 5.54 and (6.23) we have

$$\begin{aligned}\lim_{t\to\infty} t^{n/2}\,\mathrm{Vol}(X)H(p,y,t) &= \mathrm{Vol}(X)\lim_{i\to\infty} H_i(p,x,1) \\ &= \mathrm{Vol}(X)H_\infty(p_\infty,p_\infty,1) = (4\pi)^{-n/2}\frac{2\pi^{n/2}}{\Gamma(n/2)}.\end{aligned} \tag{6.29}$$

We just need to recall $n\omega_n = 2\pi^{n/2}(\Gamma(n/2))^{-1}$ (see [Ta] Chapter 3).

Finally in view of the almost rigidity theorem [ChCo1], we see the above results holds for *all* sequences $t_i \to \infty$. This suffices to complete the proof. □

Similarly, we get the asymptotic formula for heat kernels in [LiTW]:

**Corollary 6.30** (Li-Tam-Wang). *Assume $M^n$ is a complete noncompact manifold satisfying (0.4),* $\mathrm{Ric}_{M^n} \geq 0$. *Then for $p \in M^n$, and any $R, T > 0$,*

$$\lim_{d(p,x)\to\infty} \mathrm{Vol}(B_{R^{-1}d(p,x)}(p))H(p,x,Td(p,x)^2R^{-2}) = \frac{\omega_n}{(4\pi T)^{n/2}e^{R^2/4T}}. \tag{6.31}$$

*Proof.* We use the same argument as in Corollary 6.24. For $x_i$ with $d(p,x_i)\to\infty$, we study the heat kernels on the sequence $M_i^n = (M^n, p, R^2 d(p,x_i)^{-2}dx^2)$. □

We can similarly get a *local* asymptotic formula for $H_\infty$.

## 7. Eigenvalues on compact limit spaces

We assume $M_i^n \xrightarrow{d_{GH}} M_\infty$, with $\mathrm{Ric}_{M_i^n} \geq -(n-1)\Lambda$, $M_\infty$ compact. A point $x \in M_\infty$ is said to be regular, $x \in \mathcal{R}_k$, if all tangent cones at $x$ equal to $\mathbf{R}^k$; see [ChCo2].

**Lemma 7.1.** *If $x \in \mathcal{R}_n \subset M_\infty$, then*

$$\lim_{t\to 0} H_\infty(x,x,t)t^{\frac{n}{2}} = (4\pi)^{-\frac{n}{2}}. \tag{7.2}$$

*Proof.* Use a similar argument as the one in Corollary 6.24. □

**Theorem 7.3.** *Assume $M_i^n \xrightarrow{d_{GH}} M_\infty$, $\mathrm{Ric}_{M_i^n} \geq -(n-1)\Lambda$, and for some $v_0 > 0$,* $\mathrm{Vol}(M_i^n) \geq v_0$. *Then*

$$\lim_{j\to\infty} j^{-\frac{2}{n}}\lambda_{j,\infty} = 4\pi\Gamma(\frac{n}{2}+1)^{\frac{2}{n}}\mu_\infty(M_\infty)^{-\frac{2}{n}}. \tag{7.4}$$

*Proof.* In this case we don't need to renormalize the volume on $M_i^n$ (see [ChCo2]). Note for some $D$ we have $\mathrm{Diam}\, M_i^n \leq D$, $i = 1, 2, ..., \infty$, by the Bishop-Gromov inequality and (1.16), we get

$$t^{\frac{n}{2}} H_\infty(x,x,t) \leq C(n,\Lambda,D,v_0). \tag{7.5}$$

Moreover, almost every point of $M_\infty$ is in $\mathcal{R}_n$. Now by Corollary 7.1, for $x \in \mathcal{R}_n$, $t^{\frac{n}{2}}H_\infty(x,x,t) \to (4\pi)^{-n/2}$ when $t \to 0$. By the dominated convergence theorem,

$$\lim_{t\to 0} t^{\frac{n}{2}}\int_{M_\infty} H_\infty(x,x,t)dx = (4\pi)^{-\frac{n}{2}}\mu_\infty(M_\infty). \tag{7.6}$$

Finally, by applying the Karamata Tauberian theorem (see [Ta] Chapter 8), we have

$$\lim_{\lambda\to\infty} \lambda^{-\frac{n}{2}} N(\lambda) = \mu_\infty(M_\infty)\Gamma(\frac{n}{2}+1)^{-1}(4\pi)^{-\frac{n}{2}}, \tag{7.7}$$

where $N(\lambda)$ is the number of eigenvalues smaller than $\lambda$. Clearly this implies the Weyl asymptotic formula (7.4). □

When the limit space $M_\infty$ is collapsed, at present our results are less satisfactory. Recall the notion of *Minkowski dimensions*; see [Ma]. Assume $Z$ is a metric space. For $d>0$, let $N(Z,\epsilon)\in \mathbf{Z}$ be the minimal integer such that $Z$ can be covered by $N(Z,\epsilon)$ many balls of radius $\epsilon$. Put

$$v_d^-(Z) = \liminf_{\epsilon\to 0} \epsilon^d N(Z,\epsilon), \tag{7.8}$$

$$v_d^+(Z) = \limsup_{\epsilon\to 0} \epsilon^d N(Z,\epsilon). \tag{7.9}$$

Here $v_d^\pm(M_\infty)$ can be $\infty$. The upper (lower) Minkowski dimension is defined by

$$\overline{\dim}_{Mink}(Z)\ (\underline{\dim}_{Mink}(Z)) = \inf\{d | v_d^+(Z)=0\ (v_d^-(Z)=0)\}. \tag{7.10}$$

**Lemma 7.11.** *There exist $E_1(n), E_2(n)>0$ such that for any $d>0$,*

$$\limsup_{t\to 0} t^{\frac{d}{2}} \int_{M_\infty} H_\infty(x,x,t)dx \le E_2 v_d^+(M_\infty), \tag{7.12}$$

*and if, in addition,* $\mathrm{Ric}_{M_i^n}\ge 0$, *then*

$$E_1 v_d^-(M_\infty) \le \liminf_{t\to 0} t^{\frac{d}{2}} \int_{M_\infty} H_\infty(x,x,t)dx. \tag{7.13}$$

*Proof.* Let $\cup_{1\le j\le N(M_\infty,\sqrt{t})} B_{\sqrt{t}}(x_j)$ be a covering of $M_\infty$ by a minimal set of balls of radius $\sqrt{t}$. We add up the integrals of $H_\infty$ on these ball an use Corollary 2.7 to get the estimates (7.12), (7.13). □

**Lemma 7.14.** *If $v_d^+(M_\infty)<c<\infty$, then there exist $C$ such that*

$$\lambda_{j,\infty} > C j^{\frac{2}{d}}. \tag{7.15}$$

*Proof.* We can follow an argument of Gromov (see [Gr] or Theorem 4.8 in [Ch3]). Here we use the assumption $v_d^+(M_\infty)<c<\infty$ to estimate the number of balls that is needed to cover $M^\infty$. □

**Lemma 7.16.** *If $v_d^-(M_\infty)>c>0$, then there exist $C$ depending on $n, c$, such that*

$$\lambda_{j,\infty} \le C j^{\frac{2}{d}}. \tag{7.17}$$

*If $k$ is the maximal integer such that $\mathcal{R}_k\subset M_\infty$ is not empty, then*

$$\lambda_{j,\infty} < C(M_\infty)(j)^{\frac{2}{k}}. \tag{7.18}$$

*Proof.* For $r > 0$, $M_i^n$ contains $j = C(n,c)r^{-d}$ many disjoint balls of radius $r$ for $i$ big enough. The result follows by a well known argument of Cheng [Cheng]; see page 105 of [SY].

If $k$ is the maximal integer such that $\mathcal{R}_k \subset M_\infty$ is not empty, then the $k$-Hausdorff measure of $M_\infty$ is positive (see [ChCo3] or [Ch3]). So $v_k^-(X) > 0$. By (7.17) we get (7.18). □

If one can also prove for any $d > k$,

$$\lim_{t\to 0} t^{\frac{d}{2}} \sum_{j=0}^{\infty} e^{-\lambda_{j,\infty} t} = \lim_{t\to 0} t^{\frac{d}{2}} \int_{M_\infty} H_\infty(x,x,t)dx = 0, \tag{7.19}$$

then by Lemma 7.11, $d_M(M_\infty)$, the Minkowski dimension of $M_\infty$ is no more than $k$. Combine with the results in [ChCo3] and [Ch3], $d_M(M_\infty) = k$. However, at present we don't know how to get (7.19). One related question is,

*Question:* Is there an $\epsilon(n) > 0$, such that for any $M^n$ with $\mathrm{Ric}_{M^n} \geq 0$, any eigenfunction $\phi$ of $\Delta$ and any set $E$ with $\mathrm{Vol}(E) < \epsilon \,\mathrm{Vol}(M)$, we have

$$\int_{M^n - E} \phi^2 > \frac{1}{2} \int_{M^n} \phi^2 \ ? \tag{7.20}$$

## References


[Bu] P. Buser, *A note on the isoperimetric constant.* Ann. Scient. Ec. Norm. Sup. 15 (1982) 213-230.

[Ch1] J. Cheeger, *On the spectral geometry of spaces with cone-like singularities.* Proc. Nat. Acad. Sci. U.S.A. 76 (1979), no. 5, 2103–2106.

[Ch2] J. Cheeger, *Spectral geometry of singular Riemannian spaces.* J. Diff. Geom. 18 (1983), 575-657.

[Ch3] J. Cheeger, *Differentiability of Lipschitz Functions on Metric Measure Spaces.* GAFA. 9 (1999) 428-517.

[ChCo1] J. Cheeger, T. Colding, *Lower bounds on the Ricci curvature and the almost rigidity of warped products* Ann. of Math. 144 (1996) 189-237.

[ChCo2] J. Cheeger, T. Colding, *On the structure of spaces with Ricci curvature bounded below. 1* J. Diff Geom. 46 (1997), no. 3, 406–480.

[ChCo3] J. Cheeger, T. Colding, *On the structure of spaces with Ricci curvature bounded below. 2* preprint

[ChCo4] J. Cheeger, T. Colding, *On the structure of spaces with Ricci curvature bounded below. 3* preprint

[ChGr] J. Cheeger, D. Gromoll, *The splitting theorem for manifolds of nonnegative Ricci curvature.* J. Diff. Geom. 6 (1971), 119-128.

[CGT] J. Cheeger, M. Gromov, M. Taylor, *Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds.* J. Diff Geom. 17 (1982), no. 1, 15–53.

[ChTa1] J. Cheeger, M. Taylor, *On the diffraction of waves by conical singularities. I.* Comm. Pure Appl. Math. 35 (1982), no. 3, 275–331.

[ChY] J. Cheeger, S-T. Yau, *A lower bound for the heat kernel.* Comm. Pure Appl. Math. 34 (1981), no. 4, 465–480.

[Cheng] S. Y. Cheng, *Eigenvalue comparison theorems and its geometric applications.* Math. Z. 143 (1975), no. 3, 289–297.

[CY1] S. Y. Cheng, S-T. Yau, *Differential equations on Riemannian manifolds and their geometric applications.* Comm. Pure Appl. Math. 28 (1975), no. 3, 333–354.

[Co] T. Colding, *Ricci curvature and volume convergence.* Ann. of Math. (2) 145 (1997), no. 3, 477–501.

[CoMi1] T. Colding, W. Minicozzi II, *Large scale behavior of kernels of Schrdinger operators.* Amer. J. Math. 119 (1997), no. 6, 1355–1398.

[CoMi2] T. Colding, W. Minicozzi II, *Harmonic functions with polynomial growth.* J.Diff Geom. 46 (1997), no. 1, 1–77.

[CoMi3] T. Colding, W. Minicozzi II, *Harmonic functions on manifolds.* Ann. of Math. (2) 146 (1997), no. 3, 725–747.

[Fu] K. Fukaya, *Collapsing of Riemannian manifolds and eigenvalues of Laplace operator.* Invent. Math. 87 (1987), no. 3, 517–547.

[Gr] M. Gromov, *Metric structures for Riemannian and Non-Riemannian Spaces* Progress in Mathematics, 152. Birkhäuser Boston, Inc., Boston, MA, 1999.

[GT] D. Gilbarg, N. Trudinger *Elliptic Partial Differential Equations of Second Order.* Springer-Verlag, New York, 1983

[HaKo] P. Hajlasz, P. Koskela, *Sobolev meets Poincare.* C. R. Acad. Sci. Paris Ser. I Math. 320 (1995), no. 10, 1211–1215.

[Hir] M. Hirsch, *Differential Topology* Springer-Verlag, New York, 1976

[J] D. Jerison, *The Poincare inequality for vector fields satisfying Hörmander's condition.* Duke Math. J. 53 (1985), 309-338.

[KK1] A. Kasue, H. Kumura, *Spectral convergence of Riemannian manifolds.* Thoku Math. J. (2) 46 (1994), no. 2, 147–179.

[KK2] A. Kasue, H. Kumura, *Spectral convergence of Riemannian manifolds. II.* Thoku Math. J. (2) 48 (1996), no. 1, 71–120.

[Li1] P. Li, *Large time behavior of the heat equation on complete manifolds with non-negative Ricci curvature* Ann. Math. 124 (1986), 1-21.

[Li2] P. Li, *Lecture notes on geometric analysis.* Lecture Notes Series, 6. Seoul National University, Seoul, 1993.

[LiT1] P. Li, L-F. Tam, *Symmetric Green's function on complete manifolds.* Amer. J. Math. 109 (1987) 1129-1154.

[LiT2] P. Li, L-F. Tam, *Complete surfaces with finite total curvature.* J. Diff. Geom. 33 (1991) 139-168

[LiTW] P. Li, L-F. Tam, J. Wang, *Sharp bounds for the Green's function and the heat kernel.* Math. Res. Lett. 4 (1997), no. 4, 589–602.

[LiY1] P. Li, S-T. Yau, *Estimates of eigenvalues of a compact Riemannian manifold.* Geometry of the Laplace operator pp. 205–239, Proc. Sympos. Pure Math., XXXVI, AMS, Providence, R.I., 1980.

[LiY2] P. Li, S-T. Yau, *On the parabolic kernel of the Schrödinger operator.* Acta Math. 156 (1986) 153-201.

[Lin] F. Lin, *Elliptic Partial Differential Equations.* Courant Lecture Notes in Mathematics, No.1, 1997

[Ma] P. Mattila, *Geometry of sets and measures in Euclidean spaces.* Cambridge University Press, Cambridge, 1995

[Per] G. Perelman, *A complete Riemannian manifold of positive Ricci curvature with Euclidean volume growth and nonunique asymptotic cone.* Comparison geometry, 165–166, Cambridge Univ. Press, Cambridge, 1997.

[SY] R. Schoen, S-T. Yau, *Lectures on Differential Geometry.* International Press, Cambridge, MA, 1994.

[Ta] M. Taylor, *Partial Differential Equations*, Volume 1,2,3. Springer-Verlag. New York, NY, 1996

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