A GEOMETRIC INTRODUCTION TO SPACETIME AND SPECIAL RELATIVITY.

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Abstract. A narrative of special relativity meant for graduate students in mathematics or physics. The presentation builds upon the geometry of spacetime; not the explicit axioms of Einstein, which are consequences of the geometry.

1. Introduction

Einstein was deeply intuitive, and used many thought experiments to derive the behavior of relativity. Most introductions to special relativity follow this path; taking the reader down the same road Einstein travelled, using his axioms and modifying Newtonian physics. The problem with this approach is that the reader falls into the same pits that Einstein fell into. There is a large difference in the way Einstein approached relativity in 1905 versus 1912. I will use the 1912 version, a geometric spacetime approach, where the differences between Newtonian physics and relativity are encoded into the geometry of how space and time are modeled. I believe that understanding the differences in the underlying geometries gives a more direct path to understanding relativity.

Comparing Newtonian physics with relativity (the physics of Einstein), there is essentially one difference in their basic axioms, but they have far-reaching implications in how the theories describe the rules by which the world works. The difference is the treatment of time. The question, “Which is farther away from you: a ball 1 foot away from your hand right now, or a ball that is in your hand 1 minute from now?” has no answer in Newtonian physics, since there is no mechanism for contrasting spatial distance with temporal distance. In relativity, space and time are combined into **spacetime**, with an element in spacetime called an **event**. The “ball 1 foot away from your hand right now” references two events. One event is where you are right now and the other is where the ball is right now. The “ball in your hand 1 minute from now” similarly refers to two events, you right now and the ball at your hand’s location in 1 minute. The distance between events is knowable and is covered in section 3. In Newtonian physics there is only one clock and only one coordinate system is needed for space. Newtonian physics envisions space and time connected like pages in a book. Each page is space at an instant in time, and flipping the pages is time advancing. Whereas in relativity, every point in space forms its own coordinate system with its own clock which may run at a different rate from other neighboring points’ clocks, and distances from a point may be measured differently than how neighboring points measure those same distances.

This also effects how light is treated. In relativity, light is built into the geometry of spacetime in that everyone’s measurement of the speed of light is the same, and that the speed of light is the upper limit of speed that a physical particle can achieve. In Newtonian physics, there is no upper limit for the speed of a physical particle, and the notion that nothing can go faster than light has to be imposed as an artificial rule. In Newtonian physics, time is embedded in Euclidean 3-space as a parameter, whereas relativity uses a Lorentz metric (or Minkowski metric) to join time and space into spacetime, a 4-dimensional Minkowski space. This will be covered at some length in section 3.

The Lorentz transformation, and associated Lorentz metric, had been used by Lorentz, Poincaré, and others in the 1890's while exploring electrodynamics, specifically in addressing what change of coordinates would leave Maxwell's equations unchanged. Einstein in his 1905 [1] paper constructed a change of coordinates that would keep the laws of Newtonian physics intact while also preserving the speed of light for different observers, and found this was also the Lorentz transformation. With Maxwell's equations being relativistic invariants, some attention was focused in using the Lorentz transformation for measuring distances in the setting of spacetime, but it was abandoned as too strange for modeling reality. It was Minkowski in 1908, three years after Einstein published his paper on special relativity, who worked out the details showing four dimensional space with a Lorentz metric was the proper setting for special relativity. Ironically, Einstein, Lorentz, and many others, rejected out of hand the four-dimensional spacetime of Minkowski as being too complicated, and published a "more elementary" non-four-dimensional derivation of the equations for moving bodies in Euclidean space. However, it was Minkowski's four-dimensional framework that proved to be the basis for further developments in relativity. By 1912 Einstein used the Minkowski framework for his work, and it had become standard for even experimental physicists [6]. It is the Minkowski understanding of special relativity that this paper addresses.

2. Problems with the Einstein train.

One implication of the difference in how light is handled between Newtonian physics and relativity is an effect called **time dilation**. Before we tackle the geometry of spacetime, let us look at a common thought experiment based off of a thought experiment from Einstein's 1917 book on special relativity [2]. This is special relativity done in the common (but incorrect!), patched Newtonian physics way. In defense of Einstein, he never used the argument presented below. He was very careful to implement Lorentz transformations in combining different points of view.

Consider two people, one standing by the side of a railroad track and another on a train going at constant speed v along the track. In the train a light is on ceiling with a detector on the floor. Turn on the light and have both observers measure the time it takes for the light to go from the ceiling to the floor, see figures 1 and 2. Put the distances from these figures together to get the schematic in figure 3.

If Newtonian physics is correct, the times T_{out} and T_{in} are the same T, and the speeds inside and outside are different. The Newtonian computation from figure 3 is

$$\begin{aligned}(c_{out}\,T)^2 &= (c_{in}\,T)^2 + (v\,T)^2\\ c_{out} &= \sqrt{c_{in}^2 + v^2} > c_{in},\end{aligned}$$

and the speed of light is faster for the outside observer.

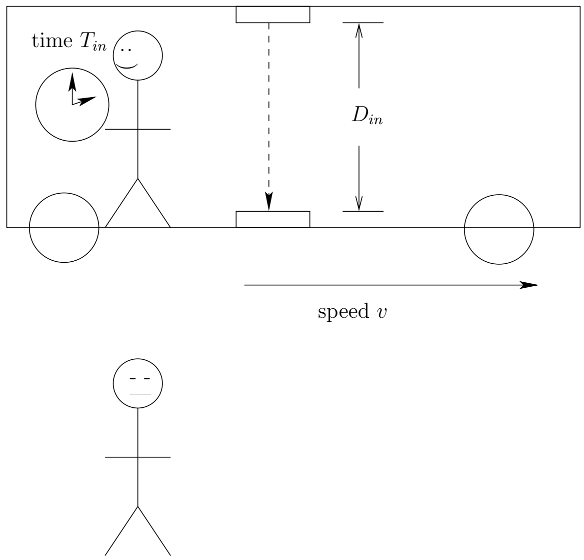

FIGURE 1. Measuring inside the train

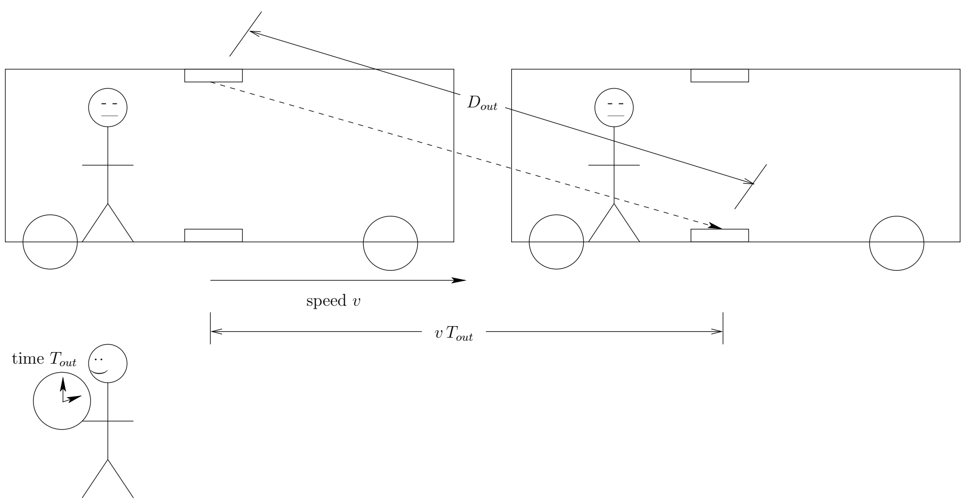

FIGURE 2. Measuring outside the train

Contrast this with the relativistic postulate that both observers measure the speed of light as c, and we have that

$$c_{in} = \frac{D_{in}}{T_{in}} = c = \frac{D_{out}}{T_{out}} = c_{out}.$$

The computation from figure 3 becomes

$$(c\,T_{out})^2 = (c\,T_{in})^2 + (v\,T_{out})^2$$
$$T_{out}\sqrt{1 - \left(\frac{v}{c}\right)^2} = T_{in}.$$

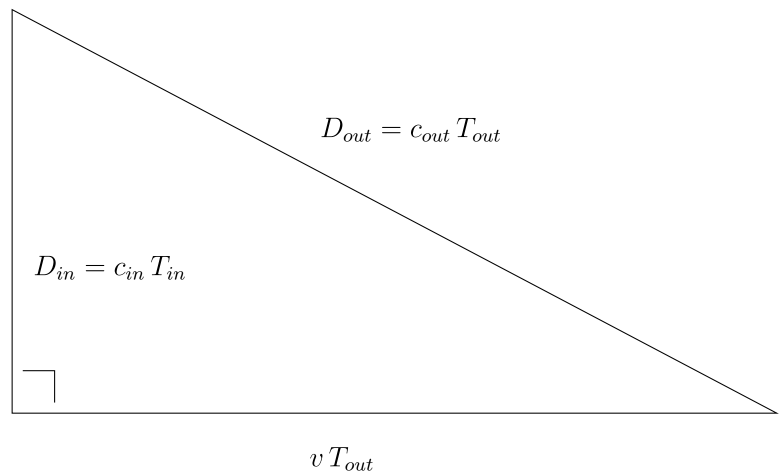

Figure 3. Combined measurements

The term $\sqrt{1-(v/c)^2} < 1$, so that $T_{out} > T_{in}$ and the clock inside the train is running slower than the clock outside the train.

The Newtonian version, that the speed of light is dependent on the observer, has been ruled out experimentally. Michelson and Morely conducted the seminal such experiment in 1887 [5]. Also, time dilation has been experimentally verified multiple times. The Hafele-Keating experiment occurred when I was a child. It captured my imagination and made me aware the world was a much stranger place than I had ever thought [3]. During October 1971, four atomic clocks were flown on jet flights around the world, eastward and westward. The time offset of the flying clocks, compared with reference clocks at the U.S. Naval Observatory, matched with the amount predicted by relativity (gravitational effects from general relativity, along with the velocity effects from special relativity).

Even though the time dilation result is correct, there is a fatal problem with this argument. One can not simply superimpose the two observer's coordinate systems as is done in Newtonian physics to get figure 3. This "superimposing" is called a Galilean transformation. Special relativity requires a Lorentz transformation to reconcile the two points of view. Before we define a Lorentz transformation, let us examine some errors that this "superimposing points of view" argument yields.

If the detector is put anywhere else, this argument will give a different result for the relationship between T_{in} and T_{out}. Move the detector forward on the floor, replacing the right angle in figure 3 by θ, see figure 4. Repeating the previous construction, the Law of Cosines applies to the triangle that combines the two points of view, figure 5, and gives

$$(c\,T_{out})^2 = (c\,T_{in})^2 + (v\,T_{out})^2 - 2\,c\,T_{in}\,v\,T_{out}\cos\theta.$$

Experimentally however, no matter where the detector is placed, the equation $(c\,T_{out})^2 = (c\,T_{in})^2 + (v\,T_{out})^2$ is the relationship for the two times, not the Law of Cosines. Indeed, in the airplane experiment, the motion does not even have to be in a straight line. The error lies in how the two viewpoints are combined into the one triangle of figures 3 and 5.

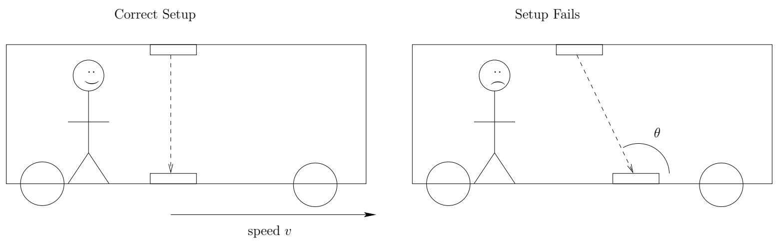

FIGURE 4. Different Detector Positions in Train

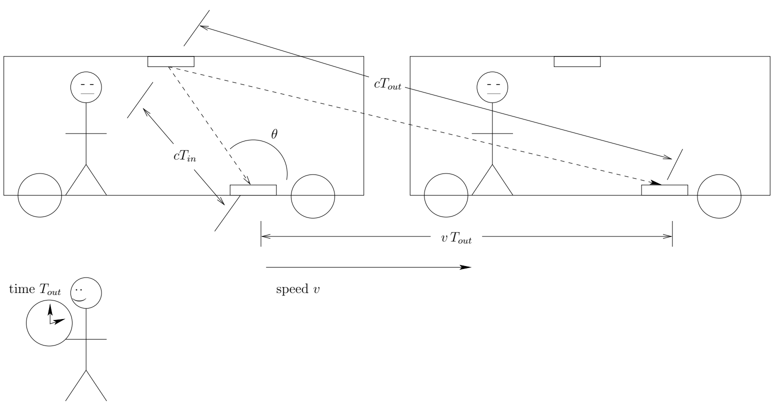

FIGURE 5. Combined Measurements: Different Detector Position

Another serious problem in the argument concerns the fact that the person inside the train believes he is stationary while the observer outside is moving. So the person in the train will measure the outside observer's clock as running slower (and indeed it is, as we will see in section 5.5), but the "superimposing points of view" construction can not reveal this fact. This is the "Simultaneity Paradox". The resolution of this paradox is, again, in combining the different points of view correctly using the geometry of spacetime.

What if instead of a light beam, we stick to what Newtonian physics is good at describing, and we throw a ball in the train? Newtonian physics and relativity both say that the outside observer measures the speed of the ball as going faster than the inside observer's measurement, but they do not agree on how much. Newtonian physics adds the velocities, relativity does not. Unfortunately for Newtonian physics, experiments confirm the speed predicted by relativity. To analyze why they don't agree and how time dilation really works, we need to set up some mathematical machinery first. We will revisit the thrown ball in section 5.6 after we set up the requisite geometry for special relativity.

3. Special Relativity

3.1. **General Relativity.** Special relativity is a special case of general relativity in the same sense that a line is a special case of a curve. Special relativity operates under the assumption that gravity has a uniform strength and direction, or equivalently, that spacetime is flat (curvature zero). This assumption suffices for many calculations by the same reasoning that calculations involving a smooth curve can be replaced by calculations on a line in a small neighborhood (the slope of a curve, for instance)

In special relativity you can move vectors around in the usual fashion and freely-falling objects move in straight lines, while in the general relativity setting you can't just move vectors around and freely-falling objects don't move in straight lines [4].

3.2. **Minkowski space: Geometry and Nomenclature.** There is one new notion that must be introduced, the Minkowski 4-space ($\mathbb{R}^4_1$), which is the structure of spacetime.

The Minkowski 4-space models the universe we live in, but does not model the world that our senses perceive. We exist in Minkowski 4-space, but our senses think we are in Euclidean 3-space with time as a parameter. The two spaces are closely related, but are quite different in their geometry. In Euclidean space, the distance between two points is measured using the Pythagorean Theorem, which is the Euclidean metric. Newtonian physics uses the Euclidean metric for space, with time as a parameter, as in the computations of section 2. Relativity uses the Minkowski metric to measure the distance between two points, or events, in spacetime.

In relativity it is usual to write the vector components as superscripts starting at zero, so for $\mathbb{R}^4$, $x = (x^0, x^1, x^2, x^3)$.

In the Euclidean case of $\mathbb{R}^4$, the length of a vector is constructed from the Euclidean inner product, the dot product, by

$$\begin{aligned}\|x\| &= \sqrt{x \cdot x} \\ &= \sqrt{(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2}.\end{aligned}$$

The length of a vector in spacetime, $\mathbb{R}^4_1$, uses the Lorentz inner product $\langle\ ,\ \rangle$ which subtracts the initial component. It is given by

$$\langle x, y\rangle = -x^0y^0 + x^1y^1 + x^2y^2 + x^3y^3.$$

The length of x in $\mathbb{R}^4_1$ is defined by

$$\begin{aligned}\|x\| &= \sqrt{|\langle x, x\rangle|} \\ &= \sqrt{|-(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2|} \\ &= \left|-(x^0)^2 + \sum_{i=1}^{3}\left(x^i\right)^2\right|^{1/2} \\ &= \left|-(x^0)^2 + |\vec{x}|^2\right|^{1/2},\end{aligned}$$

where I am using $|\cdot|$ for length of a vector in $\mathbb{R}^3$ as well as for absolute value.

The reason time is subtracted is to keep the speed of light constant for different observers. Suppose we have two observers measure a light pulse, one observer measures the pulse going distance $|\vec{x}|$ in time t, the other observer measures the pulse going distance $\left|\vec{\xi}\right|$ in time τ. The speed of light, c satisfies both equations

$$|\vec{x}|^2 = (ct)^2$$
$$\left|\vec{\xi}\right|^2 = (c\tau)^2$$

and so

$$0 = -(ct)^2 + |\vec{x}| = -(c\tau)^2 + \left|\vec{\xi}\right|^2.$$

If $x = (ct, \vec{x}) \in \mathbb{R}^4_1$ describes the passage of light then $\|x\| = 0$. A non-zero x can have "length" zero.

3.3. **Spacetime and Coordinate Systems.** An **event** is defined as a specific time and space location, where you sitting right now for instance. Spacetime is the collection of all events, and is a manifold. You can think of a manifold as a set of points with a approximate local vector space structure, just like a smooth curve is a set of points, where at each point the curve is approximately the tangent line. At each point on a smooth cure there is an approximate local vector space structure (tangent line = one dimensional vector space). A coordinate system is the function that takes as an input a point on a manifold, and gives as an output a vector in a vector space.

An event in spacetime is just an element of a set, without more information we can not know how it interacts with other events in spacetime. Einstein's two postulates are equivalent to saying that the coordinate system for the spacetime manifold gives a local $\mathbb{R}^4_1$ vector space structure; a coordinate system at an event gives the spacetime manifold a local $\mathbb{R}^4_1$ vector space structure with which that event can interact with neighboring events.

More precisely, given an event q in spacetime and a coordinate system x, the **Minkowski spacetime coordinates** of q are denoted by

$$xq = (x^0q, x^1q, x^2q, x^3q) = (x^0q, \vec{q}) \in \mathbb{R}^4_1,$$

where x^0q is the time q occurs, and $\vec{q} = (x^1q, x^2q, x^3q)$ is the spatial coordinate of q. This notation may seem odd, but it does make sense (and is standard): x is a function that acts on a point q in a manifold. The output vector xq should really be written $x(q)$, but is abbreviated like $\sin\theta$ is for $\sin(\theta)$. This structure is necessary, because every observer has their own coordinate system. In figure 6 two observers examine the event q, and use different coordinates to describe it.

Given two events in spacetime, p and q, their spacetime-distance from each other (measured by the Lorentz metric) is called the **separation** between events p and q. The separation pq is naturally defined by the length of the vector connecting them.

$$\begin{aligned}
pq &= \|\overrightarrow{pq}\| \\
&= |\langle xq - xp, xq - xp\rangle|^{1/2} \\
&= \left|-\left(x^0q - x^0p\right)^2 + \sum_{i=1}^{3}\left(x^iq - x^ip\right)^2\right|^{1/2} \qquad (1) \\
&= \left|-\left(x^0q - x^0p\right)^2 + |\vec{q} - \vec{p}|\right|^{1/2}.
\end{aligned}$$

For the Minkowski spacetime coordinate system, we will use **geometric units**, where the speed of light c and the gravitational constant G are defined as the

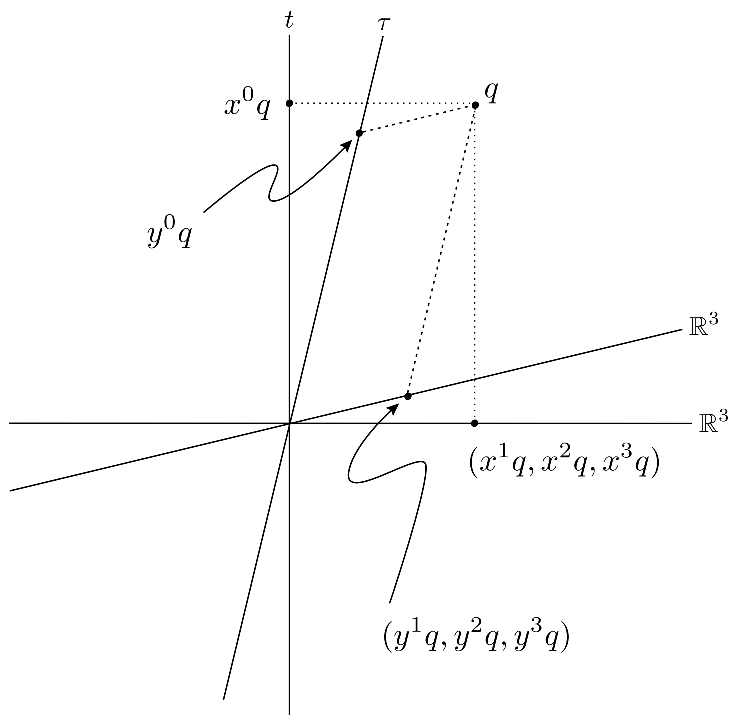

FIGURE 6. Different coordinate system views of one event q

number 1 and spacetime coordinates are made dimensionless using c and G. Only light has the same speed for every observer, so using light to measure time and distance is both natural and dimensionally consistent. For instance, one second is interpreted as one light-second, or the time it takes light to travel 3×10^{10} cm. The conversion factors from geometric units to conventional units follow from the values of c and G for that conventional system. For example in the cgs (cm, gram, second) system

$$c = 3 \times 10^{10} \ \frac{\text{cm}}{\text{sec}} \qquad G = 6.67 \times 10^{-8} \ \frac{\text{cm}^3}{\text{g}\,\text{sec}^2}.$$

The speed of light relates cm to seconds and the gravitational constant gives their relationship to grams. All measurements in spacetime can then be expressed in terms of any unit: cm, gram, or second. Let us now compare the separations of the events mentioned in the introduction: "Which is farther away from you: a ball 1 foot away from your hand right now, or a ball that is in your hand 1 minute from now?" Using $c = 58,924,800,000$ ft/min $= 1$, we can choose to measure the separation in feet,

$$\text{a ball 1 foot away from you now} = \sqrt{|-0^2 + 1^2|} = 1 \text{ ft}$$
$$\text{a ball in your hand in 1 minute} = \sqrt{|-c^2 + 0^2|} \approx 5.9 \times 10^{10} \text{ ft},$$

or in minutes

$$\text{a ball 1 foot away from you now} = \sqrt{|-0^2 + (1/c)^2|} \approx 1.7 \times 10^{-11} \text{ min}$$
$$\text{a ball in your hand in 1 minute} = \sqrt{|-1^2 + 0^2|} = 1 \text{ min.}$$

In geometric units, since length and time are expressed in terms of c, all velocities are relative to $c = 1$. To convert velocities from conventional to geometric units, divide by c and replace by the appropriate conversion. For instance 50 cm/sec is

$$\begin{aligned}\frac{50 \text{ cm/sec}}{1} &= \frac{50 \text{ cm/sec}}{c} \\ &= \frac{50 \text{ cm/sec}}{3 \times 10^{10} \text{ cm/sec}} \\ &\doteq 1.7 \times 10^{-9}\end{aligned}$$

in geometric units.

4. Measuring distance

In Euclidean space, distance is measured using the Pythagorean Theorem. The length of a line segment going from point p to q is the hypotenuse of the right triangle formed with the coordinate axes. It doesn't matter what coordinate system you use, as long as the coordinate systems are connected by a Galilean transformation (a rotation, reflection, or shift of the origin), the distance is the same. The definition of a Galilean transformation is a transformation of coordinates that preserves distance.

In $\mathbb{R}^2$, fixing the points p and q and measuring with the (x, y) or the (ξ, η) coordinate system,

$$pq = \sqrt{\Delta x^2 + \Delta y^2} = \sqrt{\Delta\xi^2 + \Delta\eta^2},$$

see figure 7. Seen from the point of view of the coordinate systems, if the point p has the same coordinates, the point q will lie somewhere on a circle of radius pq centered at p. Any Galilean transformation of coordinate systems will just put q somewhere on this circle, see figure 8.

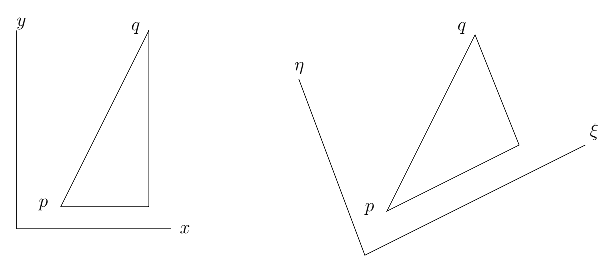

Figure 7. Measuring distance pq with different coordinates

In spacetime, where p and q now have a time component, the **Hyperbolic Pythagorean Theorem** is used to measure distance (see section 4.0.2). Form a triangle with with one leg in the time direction, one leg in the space directions, the length of the leg in space is measured using the (Euclidean) Pythagorean Theorem. The Hyperbolic Pythagorean Theorem says that hypotenuse^2 = time leg^2 − space leg^2. Similar to the Euclidean case, the distance remains the same in any coordinate system as long as the systems are connected by a suitable transformation.

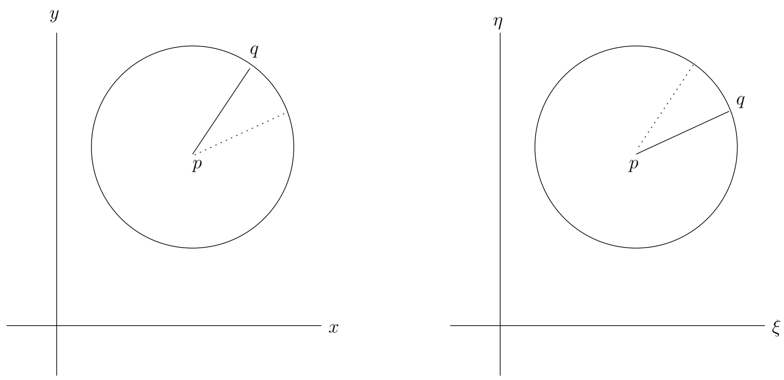

FIGURE 8. Coordinate system's view of distance pq

The collection of distance preserving transformations in spacetime are called the Lorentz transformations.

Similar to the Euclidean case of a Galilean transformation moving q along the circle, the definition of a Lorentz transformation can also be done from the point of view of coordinate systems. The point q is on a hyperbola centered at p, since for any coordinate system x,

$$pq^2 = \text{constant} = -(xq^0 - xp^0)^2 + |\vec{q} - \vec{p}|^2$$

by equation 1. Any distance preserving transformation of coordinate systems will just put q somewhere along this hyperbola. Recall from section 3.2 that this counterintuitive way of measuring distance has one redeeming feature: light is defined as lying on the line where $pq = 0$, which makes the speed of light the same for all coordinate systems linked by Lorentz transformations. If p and q are events from a light beam, then any change of coordinate system puts q on the cone centered at p, and the distance pq remains zero, see figure 10.

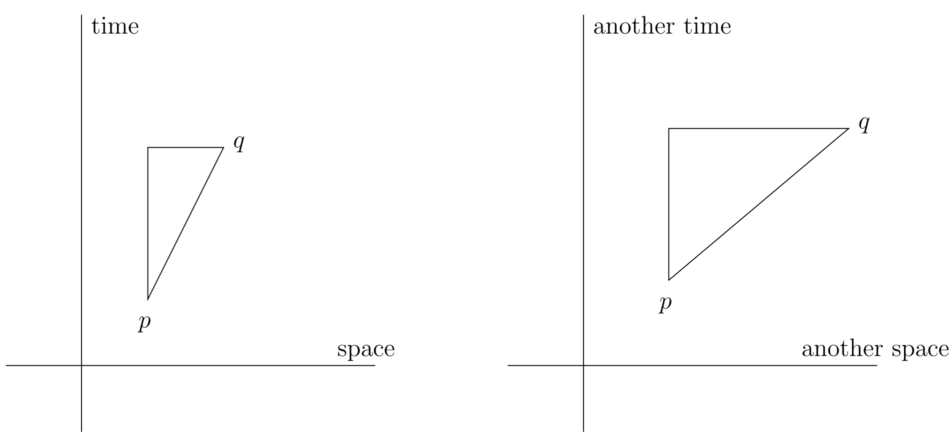

FIGURE 9. Different spacetime views of distance pq

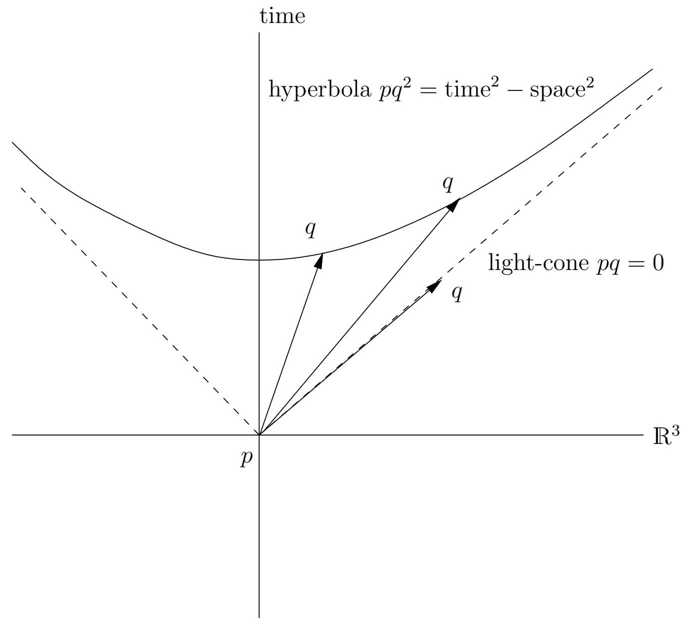

FIGURE 10. Spacetime views of distance pq plotted together

4.0.1. *Vectors in spacetime.* While in Euclidean space the set $\|\vec{pq}\| = r$ makes a sphere of radius r, in $\mathbb{R}^4_1$ it makes one of three types of hyperboloids: cone, two-sheet, or one-sheet. Rotate about the t-axis in figure 11 to visualize these hyperboloids. Note that the horizontal axis is really three dimensional Euclidean space, so these hyperboloids are actually hyperbolic three-spaces; figure 11 is just a schematic of Minkowski spacetime. If we think of o as the origin in figure 11, all events in the hyperboloid containing q are the same distance from o.

A notable consequence of the length in $\mathbb{R}^4_1$ being defined by the Lorentz metric is that some non-zero vectors can have zero length. For instance the vector (1,1,0,0) has length zero:

$$\begin{aligned}\|(1,1,0,0)\| &= \left|\Big\langle (1,1,0,0),(1,1,0,0)\Big\rangle\right|^{1/2} \\ &= \left|-(1-1)^2+(1-1)^2+0^2+0^2\right|^{1/2} = 0.\end{aligned}$$

The zero-length vectors form a cone coming out of the origin of the spacetime diagram. These vectors are called **light-like** and are the velocity vectors for light; the dashed line in figure 11 is the **light cone** for event o made up of the light-like vectors at o.

The velocity vectors for a moving physical particle are called **time-like**; these are the vectors $\vec{oq}$ with $\langle \vec{oq}, \vec{oq}\rangle < 0$. They can also go between two physical particles at different times if the particles are in the light cones of each other. The time-like vectors $\|\vec{oq}\| = r$ form a hyperboloid of two sheets. The upper sheet is the **future pointing** vectors.

Vectors $\vec{oq}$ with $\langle \vec{oq}, \vec{oq}\rangle > 0$ are called **space-like**. They normally go between two simultaneous events in space for some observer, and $\|\vec{oq}\| = r$ forms a hyperboloid of one sheet. We will see that not every observer has the same notion of

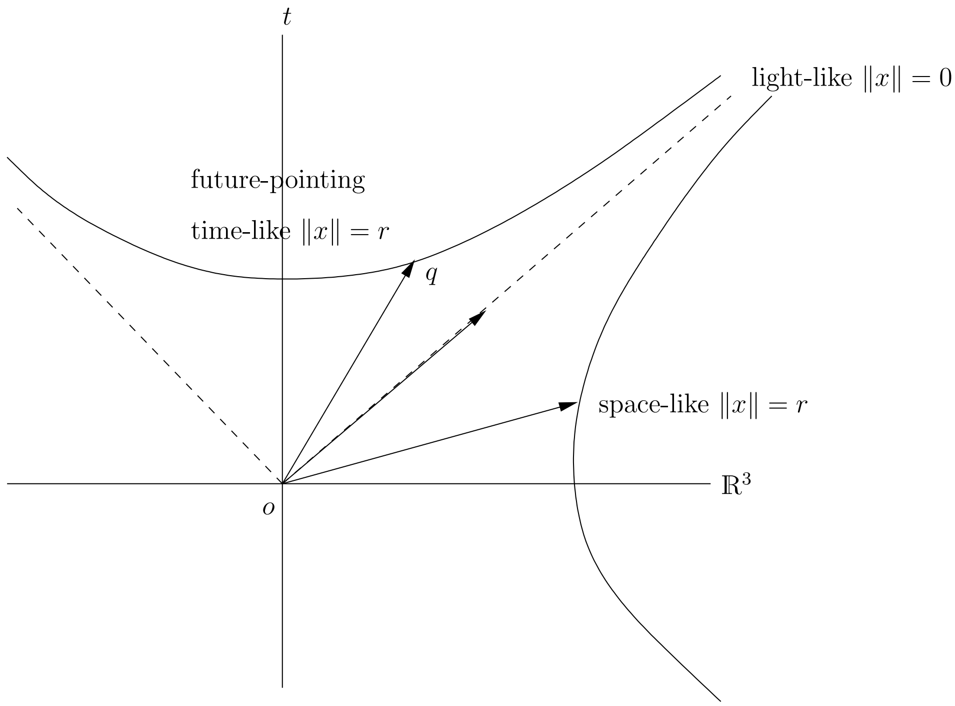

FIGURE 11. Minkowski spacetime centered at event o

simultaneous events in space, so we will need more than just horizontal vectors in the spacetime diagram to model space.

We use these hyperbolas to relate coordinate systems, reference figure 12. Take an upper sheet hyperbola that has point $(1, \vec{0})$ on it, this is one time unit from the origin. Any other point on the hyperbola is one time unit for another time axis, τ that goes through it. This defines the **proper time** for the τ-ξ coordinate system. The angle φ that the t-axis makes with the τ-axis defines the $\vec{\xi}$ space. The details are contained in the sections 4.0.2 and 4.0.3. The explicit Lorentz transformation Λ that relates the two systems, $(\tau, \vec{\xi}) = \Lambda(t, \vec{x})$, is defined in section 7.

Changing from one coordinate system to another by a Lorentz transformation is the same as making new time and space axes that follow the hyperbolas in figure 11. Light-like vectors have the same length in any coordinate system, which is 0, and any Lorentz transformation on the light cone stays in the light cone. This is the model of reality that Einstein, Lorentz, and Poincaré rejected out of hand as being too strange to be true. You are in good company if your puzzler is sore.

This definition for light, events in spacetime propagated by light-like vectors, has two important consequences. It makes all observers measure the same value for the speed of light. It also makes light a natural upper limit for the speed of physical particles, since the geometry of spacetime separates light-like vectors from the velocity vectors of physical particles, the time-like vectors. This is further defined in section 4.0.4.

To connect your physical experience with Minkowski geometry, try the following two thought experiments. Consider all the things you can theoretically see. A

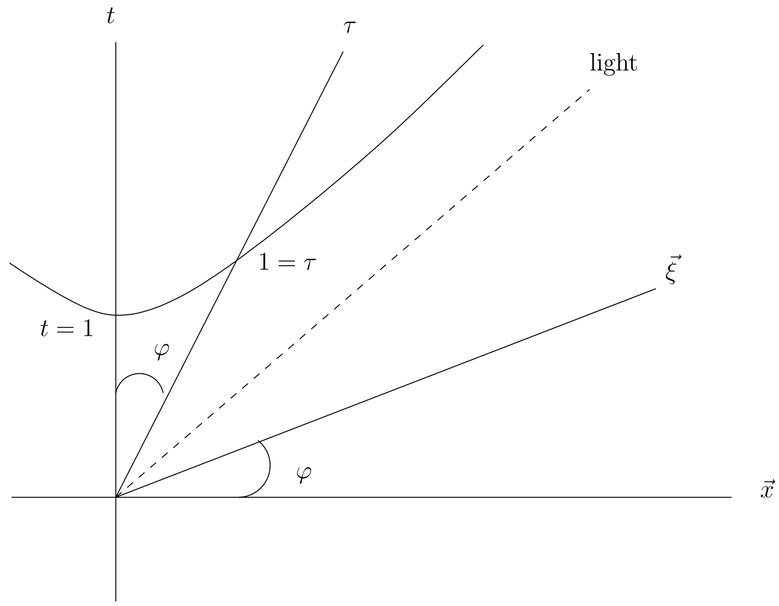

FIGURE 12. Coordinate systems

light ray connects you to it at some time in your life. Look at figure 11 and the positive t-axis is your worldline where o is your birth. The things that you can see at some time are the inside of the light-cone at o. Alternatively, every point on the positive t-axis has a separation from o, the elapsed time from your birth until then. Collecting all the events in spacetime with the same separation fills in the light cone.

Think of a horizontal plane moving up the t-axis of figure 11. The t-axis is the worldline for you sitting still and the horizontal plane is the physical space around you at that time. As the horizontal plane hits the hyperboloid containing q, what you will see is a point that turns into an expanding sphere about you. As you watch the sphere expand, while the physical points are moving away from you, the spacetime distance between your birth and the sphere remains same. The increase in time is canceling out the increase in spatial length in the Hyperbolic Pythagorean Theorem, equation (1), since the separation is constant.

4.0.2. *Hyperbolic Pythagorean Theorem.* Before we get to how Newtonian physics and special relativity fit together, we need to more precisely define the analog of the Pythagorean Theorem in $\mathbb{R}^4_1$. The Hyperbolic Pythagorean Theorem is going to be the most useful tool in analyzing common situations. Form a "right triangle" off of the time-like vector $\overrightarrow{oq}$, see figure 13.

Noting that $\overrightarrow{op} = (x^0p - x^0o, \vec{0})$ and $\overrightarrow{pq} = (0, \vec{q} - \vec{p})$ and using the definition of vector length, equation (1), we get the hyperbolic version of the Pythagorean Theorem:

$$(oq)^2 = (op)^2 - (pq)^2. \tag{2}$$

In Euclidean space, the vertical and horizontal projections of a vector are expressed using sin and cos, $\vec{x} = (r\cos\theta, r\sin\theta)$ in $\mathbb{R}^2$, and this also defines the angle

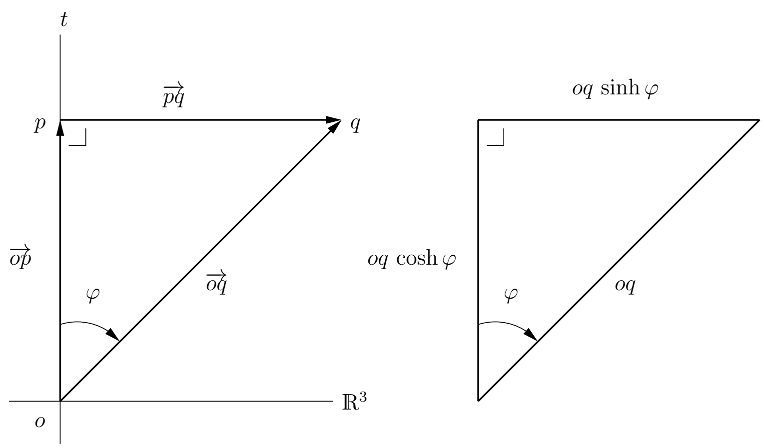

FIGURE 13. Hyperbolic Pythagorean Theorem

θ that the vector makes with the x-axis. In $\mathbb{R}^4_1$ we can use sinh and cosh similarly, since $\cosh^2 - \sinh^2 = 1$. As in figure 13,

$$pq = oq \sinh\varphi \qquad (3)$$
$$op = oq \cosh\varphi.$$

This defines the hyperbolic angle φ, the angle the vector makes with the time axis. As velocity is the change in distance over time, so it is no surprise that $\tanh\varphi$ is related to velocity. The precise relationship is defined in section 4.0.4.

The right angle symbol in figure 13 indicates that $\overrightarrow{op} \perp \overrightarrow{pq}$, which means that $\langle \overrightarrow{op}, \overrightarrow{pq} \rangle = 0$, not that there is necessarily 90° between them in the diagram. Be aware that one leg of this "right triangle", $\overrightarrow{op}$, is a time-like vector which lie on the time axis for an observer, the other leg, $\overrightarrow{pq}$, is a space-like vector which will lie in the rest space for that observer. Also, the sum of the angles in the triangle do not add up to π as in Euclidean space, since the hyperbolic angle can be as large as you want, see figure 14.

4.0.3. *Rest spaces and Worldlines.* The concept of rest space is prominent in relating special relativity to Newtonian physics, and is related to the concept of worldline. A worldline tracks a particle in spacetime, returning its spacetime coordinate in the observer's coordinate system. In Newtonian physics, to track a particle in time, time is input and a space coordinate is output; $\vec{\alpha}(t) = (\alpha^1(t), \alpha^2(t), \alpha^3(t))$. To put it in spacetime, put t as the initial component: $\alpha(t) = (t, \vec{\alpha}(t)) = (\alpha^0(t), \alpha^1(t), \alpha^2(t), \alpha^3(t))$. Given a worldline, the rest space is the three dimensional Euclidean space that is orthogonal to the time direction of the worldline.

For your worldline, your rest spaces are what you perceive around you at any instant in time. The objects in motion about you are the projections from their worldlines onto your rest space, as you are a projection onto their rest space. More precisely, given two particles (either stationary or accelerating) that trace out worldlines α and β, their time directions are their time-derviatives α' and β'. Using the coordinate system centered on α, the time for α is then along the vector $\alpha' = (1, 0, 0, 0)$ and the time for β is along some other vector $\beta' = (1, a, b, c)$. Since we are using α's coordinate system, the rest space for α is spanned by the linearly

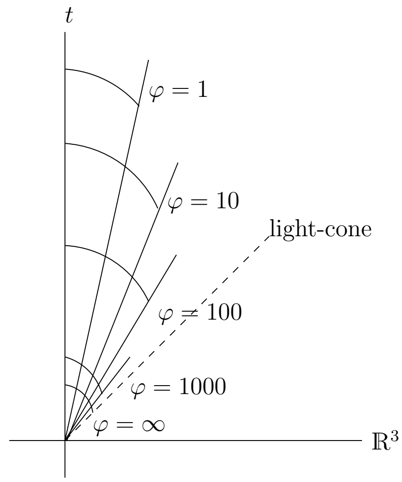

FIGURE 14. Hyperbolic angle φ

independent vectors $(0,1,0,0)$, $(0,0,1,0)$, and $(0,0,0,1)$. The rest space for β is spanned by the linearly independent vectors $(a,1,0,0)$, $(b,0,1,0)$, and $(c,0,0,1)$, since

$$\langle \beta', \text{rest space vector} \rangle = \langle\, (1,a,b,c), (c,0,0,1)\, \rangle = -c + 0 + 0 + c = 0$$

and the other two vectors are similarly orthogonal to $(1,a,b,c)$. Note that being orthogonal in spacetime is not always 90° like in Euclidean space. Using the different coordinate systems centered on α or on β the apparent angle between the time-axis and the rest space changes, see figure 15 where the rest spaces are S_α and S_β.

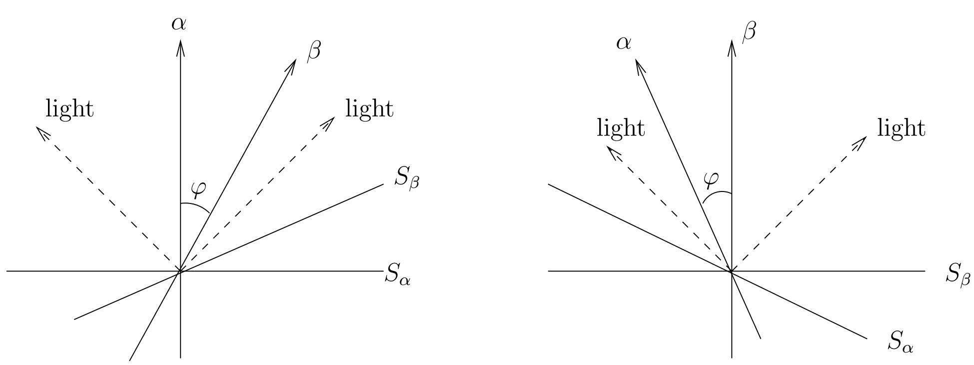

FIGURE 15. Rest spaces are orthogonal to time axes

An observation I wish to reinforce is that hyperbolic angles are not evenly distributed as angles are in Euclidean space. As vectors approach the light cone, the apparent angles between them appear more acute, see figure 16.

Another crucial definition is that each worldline has its own clock, measured by its **proper time**. The particles' proper times are the scaling needed to make

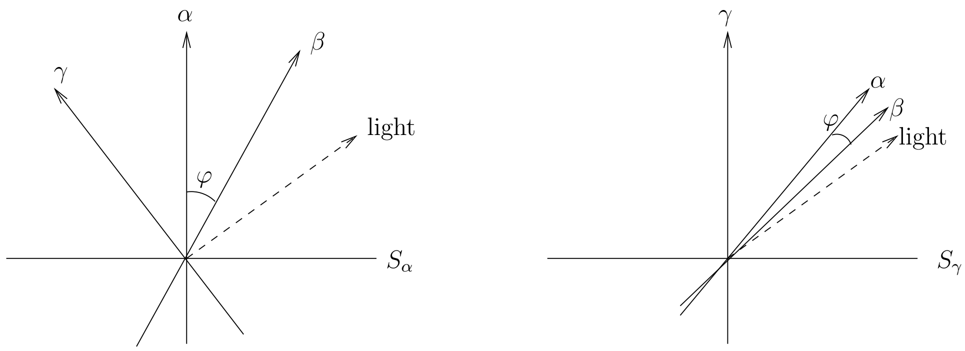

FIGURE 16. Apparent angles between worldlines α, β, and γ

$\|\alpha'\| = \|\beta'\| = 1$, so that a particle's clock never speeds up or slows down, according to the particle.

4.0.4. *Worldlines and Newtonian Physics.* A physical particle traveling through time with a future-pointing time-like derivative vector, traces out a **worldline** in spacetime. A worldline is a one parameter curve, say $\alpha(\tau)$. A moving particle in Newtonian physics is the projection of the worldline into Euclidean space. Each particle has its own clock, measured by the proper time τ so that $\|\alpha'(\tau)\| = 1$.

Referring back to figure 11, for a stationary observer at the origin, his worldline is the t-axis, $(t, \vec{0})$. Thus, the velocity vector for his motion, $(1, \vec{0})$, points straight up along the t-axis and has length 1, the height of the hyperbola in the figure. The vector $\vec{oq}$ would span the time axis for a worldline, $\alpha' = d\alpha/d\tau = \vec{oq}$ and the rest space for worldline α, which are orthogonal to the time direction of α, is spanned by the pictured space-like vector. From figure 12, the τ-axis is the time axis α' and the ξ-axis is the rest space.

To see this better, let a worldline for a particle $\alpha(\tau)$ be plotted on the coordinate system of a stationary observer, and let a particular event $\alpha(\tau_0)$ be given as in figure 17.

The t-axis is the worldline for the stationary observer and the $\mathbb{R}^3$ "axis" is the rest space for that observer at time 0. All the other rest spaces for the stationary observer stack up horizontally, one for each time. At each time t_0, the rest space intersects the worldline α once at $(t_0, \vec{\alpha}(\tau_0))$. Collapse the rest spaces together into one Euclidean 3-space, and all the $\vec{\alpha}$ points give us the familiar path in space for the particle, parameterized by τ. Now we can define how Newtonian physics and special relativity relate, referring to figure 17.

The **Newtonian particle of** α **for an observer** is defined as $\vec{\alpha}$, where x is the coordinate system of the observer and $x\alpha = (x^0\alpha, \vec{\alpha}) = (t, \vec{\alpha})$. $\vec{\alpha}(\tau_0)$ lies in the rest space for time t_0 of the observer, and is what the observer sees of α at the observer's time t_0.

Since $t = x^0\alpha(\tau)$, we can change the parameter from the proper time of the particle, τ, to the proper time of the observer, t, for finding the speed a particle has in Euclidean space. The **Newtonian velocity** of α for an observer at the time t_0 is $\frac{d\vec{\alpha}}{dt}(t_0)$, this is the velocity of the particle that the stationary observer measures with his clock using t-units for time. Some authors refer to $d\vec{\alpha}/dt$ as the velocity

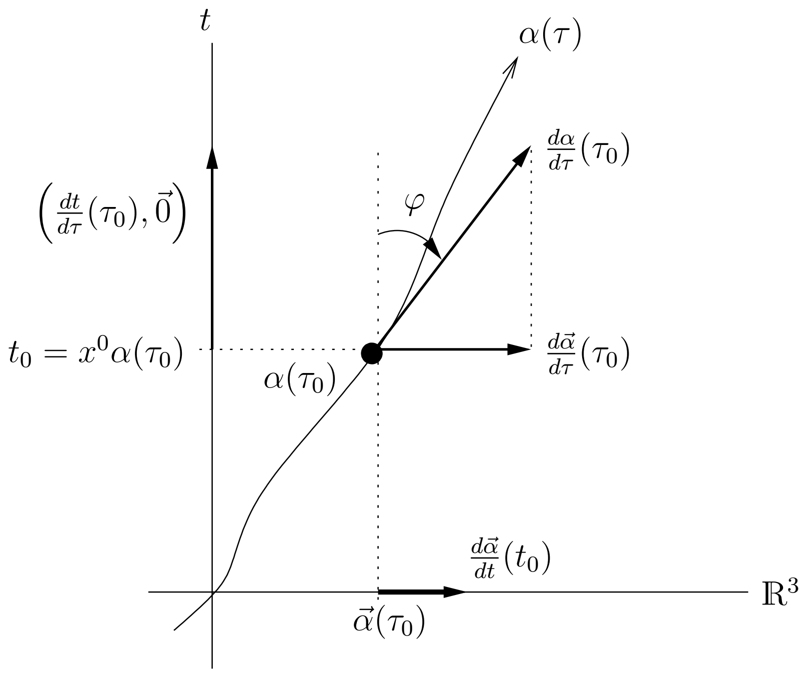

FIGURE 17. Time-like particle α

of α, and $d\alpha/d\tau$ is then called the 4-velocity. Using the Chain Rule,

$$\frac{d\vec{\alpha}}{d\tau} = \frac{d\vec{\alpha}}{dt}\frac{dt}{d\tau},$$

we can relate the velocities:

$$\frac{d\vec{\alpha}}{dt}(t_0) = \frac{d\vec{\alpha}/d\tau}{dt/d\tau}(\tau_0) = \frac{d\vec{\alpha}/d\tau}{d(x^0\alpha)/d\tau}(\tau_0). \tag{4}$$

Naturally enough, $\left|\frac{d\vec{\alpha}}{dt}(t_0)\right|$ is the speed of α relative to the observer at time t_0, or the speed of $\vec{\alpha}(t_0)$. A useful characterization is

$$\left|\frac{d\vec{\alpha}}{dt}(t_0)\right| = \tanh\varphi$$

where φ is the hyperbolic angle between the t-axis and $\frac{d\alpha}{d\tau}(\tau_0)$ (the time axes of the worldlines). This is easily seen from the Hyperbolic Pythagorean Theorem (figure 13):

$$\begin{aligned} \frac{d(x^0\alpha)}{d\tau} = \frac{dt}{d\tau} &= \left\|\frac{d\alpha}{d\tau}\right\|\cosh\varphi \\ \left|\frac{d\vec{\alpha}}{d\tau}\right| &= \left\|\frac{d\alpha}{d\tau}\right\|\sinh\varphi, \end{aligned}$$

and thus from (4),

$$\left|\frac{d\vec{\alpha}}{dt}\right| = \frac{|d\vec{\alpha}/d\tau|}{dt/d\tau} = \tanh\varphi.$$

Let us revisit the example from section 2 and put it in spacetime, see figure 18. A train moves by an outside observer at a constant rate of v mph; in 1 hour traveling v miles according to that outside observer, so in geometric units, c miles in the time direction and v miles in the rest space. Using the outside observers coordinate system $(t, \vec{x})$, plot the spacetime point (c, v). The line through the origin and that point is the time axis for the train. The angle φ is defined by $\tanh \varphi = v/c$, from the Hyperbolic Pythagorean Theorem.

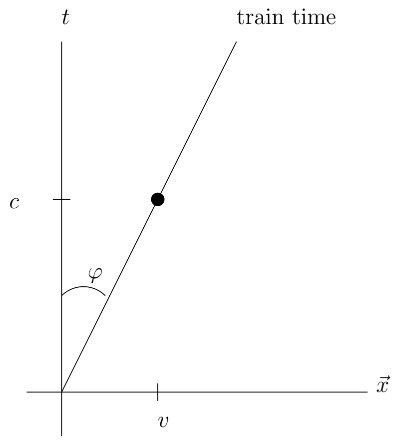

FIGURE 18. Constant Velocity

5. Examples using Special Relativity

5.1. Measuring the Speed of Light. Einstein's second postulate, the speed of light in a vacuum is the same for all observers, regardless of their relative motion or of the motion of the source of the light, is built into spacetime geometry. Pick any observer and use their coordinate system x. Let $\lambda(\tau)$ be a worldline for light, an event propagated by light-like vectors, so that $\left\|\frac{d\lambda}{d\tau}\right\| = 0$. Expressed in the x coordinate system, $x\lambda = (t, \vec{\lambda})$, we have

$$\begin{aligned}
\left\|\frac{d\,x\lambda}{d\tau}\right\|^2 &= 0 \\
-\left(\frac{d\,x^0\lambda}{d\tau}\right)^2 + \sum_{i=1}^{3}\left(\frac{d\,x^i\lambda}{d\tau}\right)^2 &= 0 \\
-\left(\frac{dt}{d\tau}\right)^2 + \left|\frac{d\vec{\lambda}}{d\tau}\right|^2 &= 0 \\
\left|\frac{d\vec{\lambda}}{d\tau}\right| &= \frac{dt}{d\tau}.
\end{aligned}$$

Thus the speed of light in any observer's coordinate system is

$$\left|\frac{d\vec{\lambda}}{dt}\right| = \frac{\left|d\vec{\lambda}/d\tau\right|}{dt/d\tau} = 1.$$

Recall that the length formula (1) is defined using geometric units, and multiplication by c converts velocities to conventional units.

5.2. **Time dilation.** Let us revisit the example from section 2 in a spacetime diagram, see figure 19. α is the worldline for the observer outside the train with proper time t and rest space S_α. β is the worldline for the lightbulb inside the train with proper time τ and rest space S_β. γ is the worldline for the detector inside the train, it also has proper time τ and rest space S_β. The speed of the train, β or γ, relative to α is v, or $v/c = \tanh\varphi$ in geometric units.

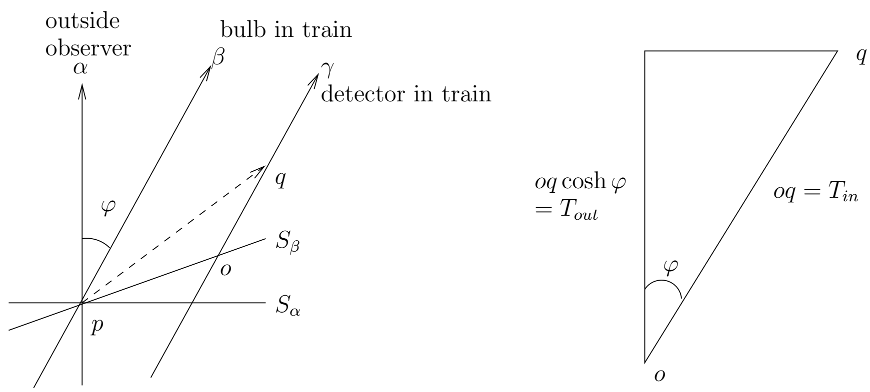

FIGURE 19. Time Dilation

The dashed line is the path of the light, where event p is the light emitted and event q is the light hitting the detector. For the detector inside the train, event o occurs at the same time as event p since they are in the same rest space. The separation oq is then the elapsed time inside the train, T_{in}. Notice the "simultaneity paradox" here: the event o does not occur at the same time as p according to the outside observer who has rest space S_α. By the Hyperbolic Pythagorean Theorem (figure 13), the time outside the train, T_{out}, is $oq\cosh\varphi$. By dividing the hyperbolic identity $1 = \cosh^2\varphi - \sinh^2\varphi$ by $\cosh^2\varphi$ we get

$$\begin{aligned} \frac{1}{\cosh^2\varphi} &= 1 - \tanh^2\varphi \\ \cosh\varphi &= \frac{1}{\sqrt{1-\tanh^2\varphi}} = \frac{1}{\sqrt{1-(v/c)^2}} \end{aligned} \tag{5}$$

and thus,

$$\begin{aligned} T_{out} &= oq\cosh\varphi \\ &= T_{in}\frac{1}{\sqrt{1-(v/c)^2}}. \end{aligned}$$

Notice that the actual location of the detector plays no role, all we need is the detector to be on the train. More directly, we can get the same result by computing the rate of change of the outside clock with respect to the inside clock: $dt/d\tau$. Using the coordinate system for α, call it x, to measure β's time and taking a τ–derivative:

$$\begin{aligned} t &= x^0\beta(\tau) \\ \frac{dt}{d\tau} &= \frac{d(x^0\beta)}{d\tau} \\ &= \left\|\frac{d\beta}{d\tau}\right\| \cosh\varphi, \text{ by (3)} \\ &= \cosh\varphi \\ &= \frac{1}{\sqrt{1-(v/c)^2}}, \text{ by (5).} \end{aligned}$$

5.3. **Space contraction.** Since time and length are related by c, and the times inside and outside the train are different, the lengths of the train as measured inside and outside the train will also be different. Using the same example as in sections 2 and 5.2, let us look at the relative lengths of the train. Figure 20 describes this situation.

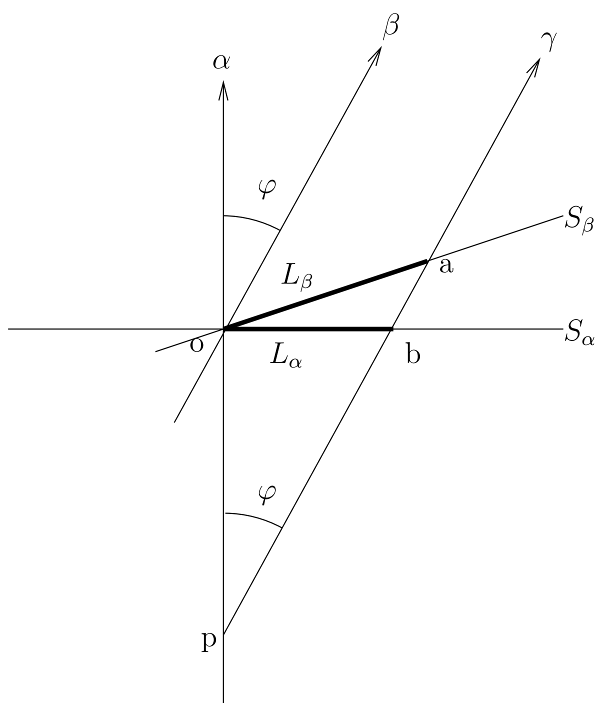

FIGURE 20. Moving Train

α is the worldline for the observer outside the train that measures the speed of the train as v, which in geometric units is $v/c = \tanh\varphi$. β is the worldline for the observer at the back of the train who believes the train is stationary. The time 0 rest space for α is S_α and is S_β for β. Event o is the back of the train at time 0 as measured by both observer's clocks, event a is the front of the train at time 0 as seen by the outside observer, and event b is the front of the train at time 0 as seen by the observer in the train. Again, notice that the events that occur at time 0 are not the same events for the inside and outside observers. Observer α then measures the length of the train as L_α and observer β measures the length of the train as L_β. We have the right triangles Δpob and Δpao, thus $L_\alpha = bp \sinh\varphi$ and $L_\beta = op \sinh\varphi$, see figure 21.

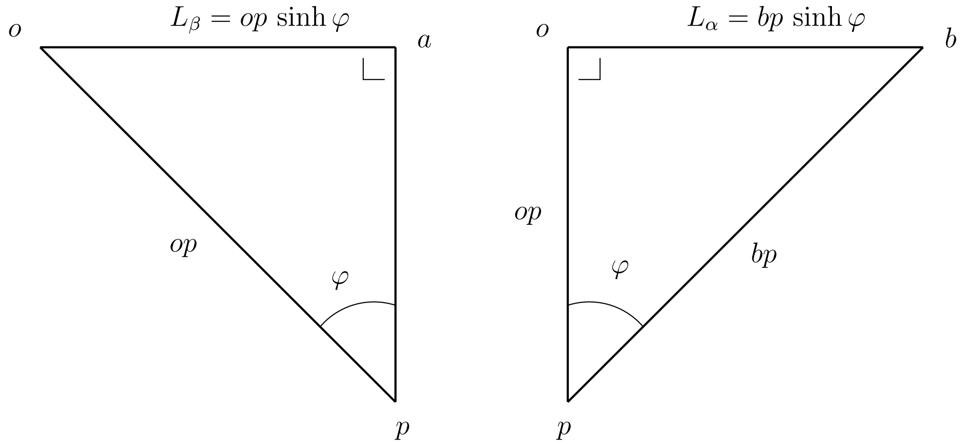

FIGURE 21. Right Triangles Δpao and Δpob

Using that $op = bp \cosh\varphi$ from the Hyperbolic Pythagorean Theorem, equation (3) or figure 13, we have

$$\begin{aligned} L_\beta &= bp \cosh\varphi \sinh\varphi \\ &= L_\alpha \cosh\varphi \\ &= \frac{L_\alpha}{\sqrt{1-(v/c)^2}}, \text{ by (5).} \end{aligned}$$

The outside observer measures the train as being smaller than the inside observer's measurement. For instance, if the observer inside the train measures it as $L_\beta = 100$ ft long, then the observer watching the train go by at $v = c\sqrt{3}/2$ ft/min thinks it is $L_\alpha = 50$ ft long. This is such a startling length difference, why don't we notice this effect? This train's speed of $v \approx 5.8$ billion miles per hour is not one we typically experience.

5.4. **Simultaneity Paradox.** Simultaneity is a statement of rest space membership. As we have seen in sections 5.2 and 5.3, events that one observer measures as simultaneous, another may not. Using the train example from those sections, put a light bulb in the center of the train and detectors equally spaced front and back from the bulb in the middle. Referring to figure 22, event o is turning on the light, p is the front detector seeing the light, q is the rear detector seeing the light. The worldlines β and γ are the detectors, and any stationary observer in the train will have a parallel worldline. The worldline α is the outside observer. Inside the train, the events p and q are simultaneous, outside they are not and the separation is a function of the angle between the worldlines. Again we see that the time direction for α and the rest spaces S_α do not make 90° in the spacetime diagram, but they are orthogonal with respect to the spacetime metric.

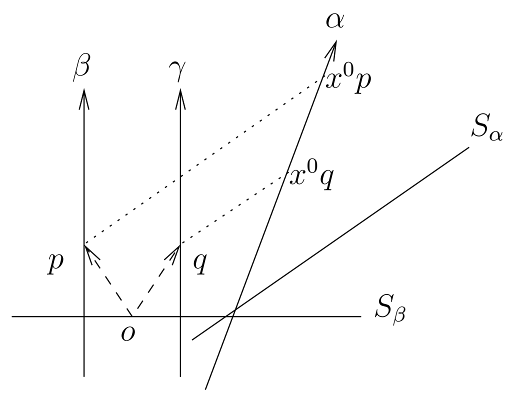

FIGURE 22. Simultaneity

5.5. **Symmetry of Time Dilation Paradox.** Given two freely-falling observers, each thinks they are stationary and the other observer is moving, so each thinks the other's clock is moving slower! The resolution of this apparent paradox is in noticing that the events used in measuring elapsed time are not symmetric. Let us look at the spacetime diagram of the situation, figure 23. In the left hand side, using the point of view of observer β, the right angle is made against the β worldline, so that by the Hyberbolic Pythagorean Theorem (figure 13)

$$\Delta\tau = \Delta t \cosh\varphi > \Delta t$$

and β thinks that α's moving clock runs slower. On the right hand side, using the point of view of observer α, the right angle is made against the α worldline and again by the Hyberbolic Pythagorean Theorem

$$\Delta t = \Delta\tau \cosh\varphi > \Delta\tau$$

and α thinks that β's moving clock runs slower. Notice that even though it looks like $\Delta\tau$ is larger than Δt, the closer a segment is to being parallel to the light cone, the smaller it is. Recall that on the light cone, the length of any segment is zero.

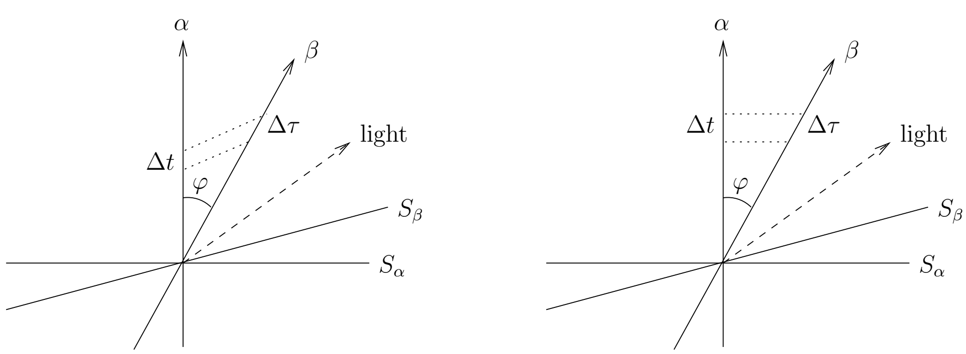

FIGURE 23. Time Perception

5.6. **Adding Velocities.** Consider the same train example as in section 2, but with a ball being thrown forward with velocity v_b by the observer inside the train. Newtonian physics says that the outside observer measures the ball's forward velocity as $v_b + v$. Look at figure 24 for the spacetime diagram of this situation.

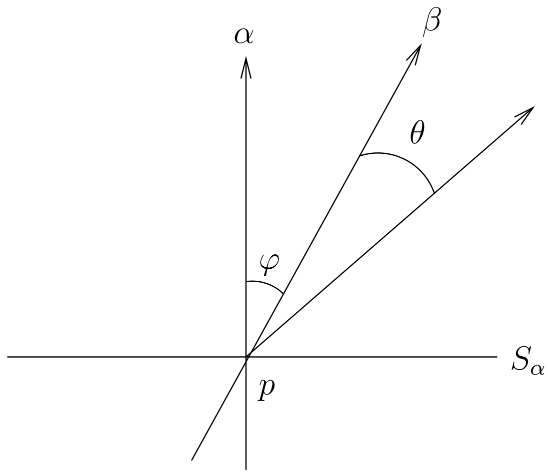

FIGURE 24. Adding Velocities

As in sections 5.2 and 5.3, α is the worldline for the outside observer that measures the train with speed v, $v/c = \tanh\varphi$ in geometric units. β is the worldline for the inside observer that measures the ball with speed v_b, $v_b/c = \tanh\theta$ in geometric units. γ is the worldline of the ball, and event p is the ball being thrown. The curve γ is not a straight line since the ball is accelerating downward, but in the forward direction it is the same motion as a ball rolled along the (frictionless) floor with speed v_b, which is a straight line. Even without that simplification, the diagram is correct for adding the velocities at the event p. From figure 24 one can see that **adding the Newtonian velocities is adding the hyperbolic angles** made with the time axes in spacetime. The outside observer measures the ball's speed in geometric units as $\tanh(\varphi + \theta)$, and using the hyberbolic tanh addition identity

$$\begin{aligned}\tanh(\varphi+\theta) &= \frac{\tanh\varphi + \tanh\theta}{1+\tanh\varphi\,\tanh\theta}\\ &= \frac{\frac{v}{c}+\frac{v_b}{c}}{1+\frac{v}{c}\,\frac{v_b}{c}}.\end{aligned}$$

Since v and v_b are small relative to c, $\tanh(\varphi + \theta) \cong (v + v_b)/c$ and the outside observer measures the ball going approximately $v + v_b$ in conventional units. This approximation is usually good enough. To illustrate, if $v = 60$ mph and $v_b = 50$

mph then the outside observer measures the ball's speed in mph as

$$\begin{aligned}\tanh(\varphi+\theta)\,c &= \frac{\frac{60}{c}+\frac{50}{c}}{1+\frac{60}{c}\frac{50}{c}}\,c \\ &= \frac{60+50}{1+60\cdot 50/c^2}\text{ mph.} \\ &\approx \frac{110}{1+6.7\cdot 10^{-15}}\text{ mph} \\ &\approx 110\text{ mph.}\end{aligned}$$

5.7. **Twin Paradox.** One twin stays home, the vertical worldline in figure 25. The other twin takes off with speed $v = c\tanh\varphi$ as measured by the twin who stayed home, turns around and comes back. The twin who stayed home is now older than the twin that took off and came back. There is no paradox. Relative to the events "leave" and "arrive" in figure 25, moving clocks run slower. Refer to example 5.2. Alternatively, notice that the separation between the events "leave" and "turn" is half the elapsed time for the twin at home, and apply the analysis of section 5.5. Either way, the result is $T_{home} = T_{away}\cosh\varphi = T_{away}/\sqrt{1-(v/c)^2}$.

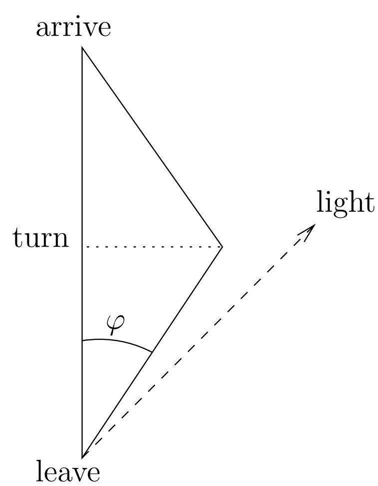

FIGURE 25. Twin Paradox

6. Energy-Momentum

Just as space and time are linked, momentum and energy are also linked. If $\beta = (\beta^0, \beta^1, \beta^2, \beta^3) = (\beta^0, \vec{\beta})$ is the worldline for a material particle of mass m, the **energy-momentum vector field** for β is $m\,\beta'$, where the derivative is with respect to β's proper time. The spatial component of the particle, $m\vec{\beta}'$, is its momentum, and the time component is the total energy, $m(\beta^0)' = E$. Why is energy in the time component? Einstein wanted to preserve the principle of conservation of momentum, and this is what balanced the equation.

Using the results from section 5.2, if β has proper time τ, then any other observer that measures the particle's velocity as v with their proper time t will measure

$$m\frac{d\beta}{d\tau} = m\frac{d\beta}{dt}\frac{dt}{d\tau} = \frac{m}{\sqrt{1-(v/c)^2}}\frac{d\beta}{dt}$$

and will conclude that the particle has more mass; the mass going to infinity as the velocity goes to the speed of light. This is the misleading concept of "relativistic mass", founded in the Newtonian thinking that mass is same for all observers, coming from a counting of protons and neutrons. A better interpretation is that the observer measures the particle as having more energy than the particle measures for itself. The relativistic interpretation is that momentum is linked with mass like space is linked with time, it is not meaningful to talk of one without the other.

However, there is a backwards compatibility in the new mass definition. The "rest mass" or "invariant mass" of a particle is the mass of a particle moving without acceleration using the particle's own coordinate system, and is the familiar counting of protons and neutrons. If the particle is not accelerating, then from the particle's point of view it is stationary. Using the particle's coordinate system, its worldline is $\beta = (\tau, 0, 0, 0)$ with $m\beta' = (m, \vec{0})$. The total energy of the particle, E, is only potential energy. $E = m$ in geometric units, and converting to conventional units gives the celebrated equation $E = mc^2$.

7. Time Dilation, Length Contraction, and Simultaneity explicitly computed with Lorentz Transformations

Remember that a linear change of coordinates can be expressed as a matrix, with the columns of the matrix being the image of the basis vectors. This does not depend on the underlying metric used to measure the size of the grid generated by the basis; it is a direct property of the definition of a vector space. The explicit Lorentz transformation for changing from the α coordinate system $(t, \vec{x})$ to the β coordinate system $\left(\tau, \vec{\xi}\right)$ in section 5.3, $\Lambda \begin{bmatrix} t \\ \vec{x} \end{bmatrix} = \begin{bmatrix} \tau \\ \vec{\xi} \end{bmatrix}$, is

$$\Lambda = \begin{bmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The outside observer has back bumper with coordinate $(t, x^1, x^2, x^3) = (0, 0, 0, 0)$ and the top of the train at the back bumper $(t, x^1, x^2, x^3) = (0, 0, h, 0)$. After time t with velocity v in the x^1-direction measured by the outside observer, the coordinates of the trains' ends are $(t, vt, 0, 0)$ and $(t, vt, h, 0)$. The outside observer measures the front of the train similarly with coordinates $(0, L_\alpha, 0, 0)$ and $(0, L_\alpha, h, 0)$, and at time t with coordinates $(t, vt + L_\alpha, 0, 0)$ and $(t, vt + L_\alpha, h, 0)$, see figure 26.

Using that $v = \tanh\varphi$,

$$\cosh\phi - v\sinh\phi = 1/\cosh\phi,$$

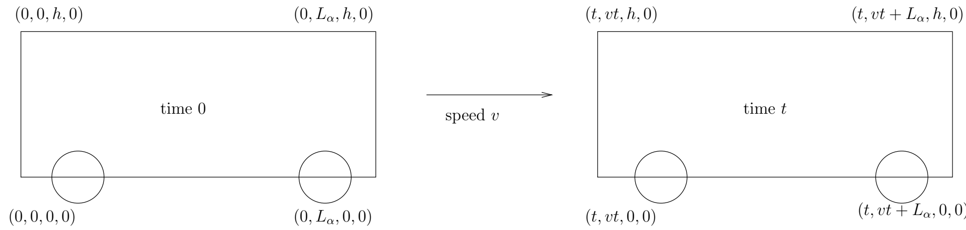

Figure 26. Train Coordinates, outside observer

and direct calculation on the coordinates for the back of the train at time 0

$$\Lambda \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \Lambda \begin{bmatrix} 0 \\ 0 \\ h \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ h \\ 0 \end{bmatrix}$$

and at time t

$$\Lambda \begin{bmatrix} t \\ vt \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} t/\cosh\phi \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \tau \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \Lambda \begin{bmatrix} t \\ vt \\ h \\ 0 \end{bmatrix} = \begin{bmatrix} t/\cosh\phi \\ 0 \\ h \\ 0 \end{bmatrix} = \begin{bmatrix} \tau \\ 0 \\ h \\ 0 \end{bmatrix}$$

shows time dilation, where t maps to $\tau = t/\cosh\varphi = t\sqrt{1-v^2}$, with no change in the height of the train between observers. Calculating for the front of the train at time 0,

$$\Lambda \begin{bmatrix} 0 \\ L_\alpha \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -L_\alpha \sinh\phi \\ L_\alpha \cosh\phi \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -L_\alpha \sinh\phi \\ L_\beta \\ 0 \\ 0 \end{bmatrix} \quad \Lambda \begin{bmatrix} 0 \\ L_\alpha \\ h \\ 0 \end{bmatrix} = \begin{bmatrix} -L_\alpha \sinh\phi \\ L_\alpha \cosh\phi \\ h \\ 0 \end{bmatrix} = \begin{bmatrix} -L_\alpha \sinh\phi \\ L_\beta \\ h \\ 0 \end{bmatrix}$$

and at time t,

$$\Lambda \begin{bmatrix} t \\ vt + L_\alpha \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \tau - L_\alpha \sinh\phi \\ L_\beta \\ 0 \\ 0 \end{bmatrix} \qquad \Lambda \begin{bmatrix} t \\ vt + L_\alpha \\ h \\ 0 \end{bmatrix} = \begin{bmatrix} \tau - L_\alpha \sinh\phi \\ L_\beta \\ h \\ 0 \end{bmatrix}$$

again shows time dilation, non-simultaneity in the time for the front of the train, and length contraction in the direction of motion, where the front of the train, L_α maps to $L_\beta = L_\alpha \cosh\varphi = L_\alpha/\sqrt{1-v^2}$.

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