Topology Comprehensive Exam

Name:____________________

Instructions: Answer six (6) questions total. On the first page of the exam, please write the numbers for the problems that you want graded.

Please present your solutions in an orderly manner and with appropriate detail. Label each submission page with a problem number and page number (i.e. "Problem 3, Page 2"). Keep all writing at least 1 inch away from page boundary. **Failure to adhere to these guidelines may result in the loss of credit.**

Problem							Total
Score							

1.

(a) Given two topologies $\mathcal{T}$ and $\mathcal{T}'$, define the product topology $\mathcal{T} \times \mathcal{T}'$.

(b) Show that the dictionary order topology on $\mathbb{R}_s \times \mathbb{R}_s$ is the same as the product topology on $\mathbb{R}_d \times \mathbb{R}_s$, where $\mathbb{R}_d$ denotes $\mathbb{R}$ with the discrete topology and $\mathbb{R}_s$ denotes $\mathbb{R}$ with the standard topology.

(c) Compare $\mathbb{R}_d \times \mathbb{R}_s$ to the standard topology on $\mathbb{R} \times \mathbb{R}$.

2. Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are "eventually zero," that is, all sequences $(x_1, x_2, \dots)$ such that $x_i \neq 0$ for only finitely many values of i.

(a) Define the box topology on $\mathbb{R}^\omega$

(b) Define the product topology on $\mathbb{R}^\omega$

(c) What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the box topology? Justify your answer.

(d) What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the product topology? Justify your answer.

3

(a) Define what it means for a topological space to be compact.

(b) Show that if Y is compact, then the projection $\pi_1 : X \times Y \to X$ is a closed map.

(c) Given an example of topological spaces X and Y (not necessarily compact), such that the projection map $\pi_1 : X \times Y \to X$ is not a closed map

4 Let $f : X \to Y$ be continuous. Mark each of the following statements true or false. If you mark a statement as false, you must provide a counter example.

(a) If X is Lindelöf, so is its image $f(X)$

(b) If X has a countable dense subset, then so does its image $f(X)$.

(c) If X is connected, so is $f(X)$

(d) if X is 1st countable, so is $f(X)$.

(e) if X is Hausdorff, so is $f(X)$.

5 Assume X is compact, Hausdorff. Assume there is a sequence of compact subsets $A_1 \supset A_2 \supset A_3 ...$ in X.

(a) If for all n, $\bigcap_{j=1}^{n} A_j \neq \emptyset$, prove that $\bigcap_{j=1}^{\infty} A_j \neq \emptyset$.

(b) Assume U is an open set so that $\bigcap_{j=1}^{\infty} A_j \subset U$. Prove that for sufficiently large n, $A_n \subset U$. [hint: find an open cover of $X - U$.]

6

(a) Given a metric space (X, d) and a set W in X, prove that the function $d(x, W) : X \to \mathbb{R}$ where $d(x, W) = inf_{w \in W}\, d(x, w)$ is continuous. You may use standard facts about metric spaces in your proof.

(b) Assume U is an open set in the 3-dimensional space $\mathbb{R}^3$. Prove that there is a sequence of compact sets

$$K_1 \subset K_2 \subset K_3 \subset \dots\dots$$

such that $U = K_1 \cup K_2 \cup K_3 \cup \dots\dots.$

7 Assume $X_1, X_2, X_3, ...$ is a sequence of topological spaces, $Y_1, Y_2, Y_3, ...$ is a sequence of topological spaces. Assume for each $j = 1, 2, 3, ...$, there is a map $f_j : X_j \to Y_j$. Define

$$f : \prod_{j=1}^{\infty} X_j \longrightarrow \prod_{j=1}^{\infty} Y_j$$

, where both the domain and range are equiped with the product topology by,

$$f(x_1, x_2, x_3, ...) = \big(f_1(x_1), f_2(x_2), f_3(x_3), ...\big).$$

Prove that f is continuous if and only if every f_j is continuous.

8 Let S be the set of real numbers with a topology τ, such that the map $f(x, y) = x + y$ is a continuous map from $S \times S$ with the product topology to S, and $g(x) = -x$ is a continuous map from S to S.

(a) Assume $a \in S$. Prove that U is open iff $a + U = \{a + x | x \in S\}$ is open.

(b) Assume U is a neighborhood of $0 \in S$. Prove that there is a neighborhood W of 0 such that $W + W = \{x + y \,|\, x, y \in W\}$ is included in U, and $W - W = \{x - y \,|\, x, y \in W\}$ is included in U.

(c) Assume the topology τ is T_1. Prove that τ is Hausdorff.

9 Suppose X is a T_1 topological space. Prove X is T_3 **if and only if** for every point $x \in X$ and every neighborhood U of x there exists a neighborhood V of x such that $x \in V \subset \overline{V} \subset U$.

10

(a) Define quotient map.

(b) Given a topological space X with an equivalence relation $\sim$. Define the quotient topology on the set of equivalence classes $X/\sim$.

(c) Define an equivalence relation on $X = \mathbb{R}$ with the standard topology by setting

$$x \sim y \quad \text{if} \quad x - y \in \mathbb{Q}.$$

Give the familiar space to which $X/\sim$ is homeomorphic. Carefully prove your answer.