Topology Comprehensive Exam

Name:______________________

Answer six (6) questions total. On the first page of your work, please write the numbers of the problems that you want graded. On each page please write only on the front side.

Problems							Total
Scores							

1.
a. Define the *discrete topology* on a set X.

b. Define a T_1-space.

c. Assume X is a T_1-space, and for any family of open subsets, $\{A_\lambda\}_{\lambda \in \Lambda}$, the intersection $\bigcap_{\lambda \in \Lambda} A_\lambda$ is also open. Prove that the topology of X is the discrete topology

2.
a. Give the definition of the *closure* $\overline{A}$ of a subset $A \subseteq X$.

b. Let X and Y be topological spaces and let $A \subseteq X$. Prove that if $f : X \to Y$ is continuous, then $f(\overline{A}) \subseteq \overline{f(A)}$.

c. Let X be a Hausdorff space, and let $A \subseteq X$. Suppose there exists $x \in \overline{A}$ such that $x \notin A$. Prove that every open set that contains x must contain infinitely many points of A.

3.
a. Let X be a topological space. Give the definition of a dense subset of X.

b. Assume U is an open subset of X, and A is a dense subset of X. Prove that $U \subseteq \overline{A \cap U}$, where $\overline{A \cap U}$ denotes the closure of $A \cap U$ in X.

4.
a. Define what it means for X to be *connected* and for X to be *path connected.*

b. Prove that the continuous image of a connected space is connected.

c. Give an example of a space that is connected but not path connected. You do not need to justify your answer.

5.
a. Define what it means for a topological space X to be *normal.*

b. Prove that a compact subset of a T_2 space is closed.

c. Prove that a compact T_2 space is normal.

6. If X is an infinite set, then the *cofinite topology* on X consists of the collection of subsets of X whose complement is finite, together with the empty set.

a. Prove that the cofinite topology on X is indeed a topology.

b. Prove that $\mathbb{R}$ with the cofinite topology is connected.

c. Is $\mathbb{R}$ with the cofinite topology Hausdorff? Justify your answer.

d. Prove that $\mathbb{R}$ with the cofinite topology is compact.

7. a. Let X, Y, and Z be topological spaces. Prove that if $f : X \to Y$ is continuous and $g : Y \to Z$ is continuous, then $g \circ f : X \to Z$ is continuous.

b. Let $X = \prod_{\alpha \in A} X_\alpha$ be a product of topological spaces and let Y be a topological space. Let $\pi_\alpha : X \to X_\alpha$ be the projection map to the factor with index α. Prove that a function $f : Y \to X$ is continuous if and only if $\pi_\alpha \circ f : Y \to X_\alpha$ is continuous for each $\alpha \in A$.

8. a. Prove that if X is a compact space and Y is a Hausdorff space, and if $f : X \to Y$ is continuous, one-one, and onto, then f is a homeomorphism.

b. Prove that if $f(X, \tau)$ is a compact Hausdorff space and τ' is (strictly) weaker than τ then (X, τ') is not Hausdorff.

c. Prove that if (X, τ) is a compact Hausdorff space and τ' is (strictly) stronger than τ, then (X, τ') is not compact.

9. For each of the collection of properties below, give an example of a topological space with those properties. You do not need to justify your answers.

a. T_1 and not T_2

b. T_2 and not T_3

c. T_3 and not T_4

d. Lindelöf and not 2nd-countable

e. First countable and not 2nd-countable

10. a. Define *quotient map.*

b. Given an equivalence relation $\sim$ on a topological space X, define the quotient topology on the set of equivalence classes $X/\sim$.

c. Let $X = \mathbb{R}^3 - \{\mathbf{0}\}$ with the subspace topology inherited from the product topology on $\mathbb{R}^3$. Define an equivalence relation $\sim$ on X given by $\mathbf{x} \sim \mathbf{y}$ if there exists a positive constant λ such that $\mathbf{x} = \lambda\mathbf{y}$. Give the well-known topological space to which $X/\sim$ with the quotient topology is homeomorphic.