Topology Comprehensive Exam

Name:____________________

Answer six (6) questions total. On the first page of your work, please write the numbers of the problems that you want graded. On each page please write only on the front side.

1.

a. Prove that every path-connected topological space is connected. You may appeal to standard results in your proof as long as you do not trivialize the problem.

b. Give an example of a topological space that is connected, but not path-connected. Prove your space is connected. Prove your space is not path-connected. You may appeal to standard results in your proofs as long as you do not trivialize the problem.

2.

a. Define an equivalence relation $\sim$ on $\mathbb{R}$ with the standard topology so that the quotient space $\mathbb{R}/\sim$ is homeomorphic to S^1 with the standard topology.

b. Define an equivalence relation $\sim$ on $\mathbb{R}^2$ with the standard product topology so that the quotient space $\mathbb{R}^2/\sim$ is homeomorphic to $S^1 \times S^1$ with the standard product topology.

c. Given $A \subset X$ with the subspace topology, a retraction is a continuous map $r : X \to A$ such that $r(a) = a$ for all $a \in A$. Prove that every retraction is a quotient map.

3.

a. Define basis for a topology on a set X.

b. Define topology generated by a basis $\mathcal{B}$.

c. Suppose that τ is a topology on a set X. Let $\mathcal{C} \subset \tau$ such that for every $U \in \tau$ and for every $x \in U$ there exists $C \in \mathcal{C}$ such that $x \in C \subset U$. Prove that $\mathcal{C}$ is a basis.

d. Given the same set up as part c, prove that the topology generated by the basis $\mathcal{C}$ is the original topology τ.

4.

a. What is a T_1 space?

b. Assume X is a T_1 space that containing at least two points. Assume X is connected. Prove that X contains infinitely many points.

5. Assume X contains infinitely many points, and $U \subset X$ is open iff $X - U$ is finite or $U = \emptyset$.

a. Prove that X is compact.

b. Prove that X is connected.

c. Prove that X is separable.

d. Is X always second countable? Why?

6.

a. Assume A is an open subset of X. Prove that for any subset B of X we have
$$A \cap \overline{B} \subset \overline{A \cap B}.$$
Here for any subset S, $\overline{S}$ means the closure of S.

b. Give an example that A is not open and $A \cap \overline{B} \subset \overline{A \cap B}$ is not true.

7. Let $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in J}$ be a collection of topological spaces. Let $X = \prod_{\alpha \in J} X_\alpha$ be their cartesian product. For each $\alpha \in J$, let $\pi_\alpha : X \to X_\alpha$ be the projection map. Let τ be the smallest topology on X, such that each π_α is continuous. Show that τ is the product topology on X.

8.

a. Let X be a Hausdorff space. Let $K_1 \supset K_2 \supset K_3 ...$ be a sequence of compact subsets of X. Assume U is an open set containing the intersection of all K_j. Prove that $K_n \subset U$ for sufficiently large n.

b. Let X be a Hausdorff space. Let $K_1 \supset K_2 \supset K_3 ...$ be a sequence of compact subsets of X. Assume that every K_j is connected. Prove that $\bigcap_{j=1}^{\infty} K_j$ is connected.

c. Give an example of a Hausdorff space X with $F_1 \supset F_2 \supset ...$ a sequence of closed subsets of X, such that every F_j is connected, yet $\bigcap_{j=1}^{\infty} F_j$ is not connected.

9.

a. Define the product topology on $\mathbb{R}^\omega$

b. Define the box topology on $\mathbb{R}^\omega$

c. Given a topology τ on $\mathbb{R}^\omega$, define what it means for a sequence $\vec{x}_n$ in $\mathbb{R}^\omega$ to converge to an element $\vec{y}$ of $\mathbb{R}^\omega$.

d. Prove that if $\vec{x}_n$ is a sequence in the box topology on $\mathbb{R}^\omega$ that converges to $\vec{y}$, then $\vec{x}_n$ converges to $\vec{y}$ in the product topology.

e. Give an example of a sequence $\vec{x}_n$ of $\mathbb{R}^\omega$ and an element $\vec{y}$ such that $\vec{x}_n$ converges to $\vec{y}$ in the product topology but not in the box topology.

10. Let X be a metrizable space with metric d.

a. Let A be a closed subset of X. Define $g : X \to \mathbb{R}$ by $g(x) = \inf_{a \in A} d(a, x)$. Prove that g is continuous.

b. Show that X is completely regular, that is X is Hausdorff and for any closed set A of X and point $b \in X \setminus A$, there is a continuous function $f : X \to \mathbb{R}$, such that $f(x) = 0$ for all $x \in A$ and $f(b) = 1$. You may appeal to standard results in your proof as long as you do not trivialize the problem. Suggestion: use part a.

c. A subset Z of X is called a *zero-set* if there exists a continuous function $f : X \to \mathbb{R}$, such that $f(x) = 0$ for all $x \in Z$. Show that every closed set of X is an intersection of zero-sets. Suggestion: use part b.