Topology Comprehensive Exam

Name:____________________

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Problem							Total
Score							

1. Let τ_α for each $\alpha \in A$ be a topology on $\mathbb{R}$.

a. Prove or provide a counter example to the statment: $\bigcup_{\alpha\in A} \tau_\alpha$ is a topology on $\mathbb{R}$.

b. Prove or provide a counter example to the statment: $\bigcap_{\alpha\in A} \tau_\alpha$ is a topology on $\mathbb{R}$.

2. A basis of closed sets for a topology τ on a set X is a collection of closed sets $\mathcal{B}$ in X such that any closed set in X is an (potentially arbitrary) intersection of elements of $\mathcal{B}$.

a. Show that $\mathcal{B}$ is a basis of closed sets for X if and only if $\{X - B \mid B \in \mathcal{B}\}$ is a basis for τ.

b. Show that $\mathcal{B}$ is a basis of closed sets for τ if and only if whenever $x \in X$ and F is closed in X *such that x is not an element of F*, there is an element B of $\mathcal{B}$ that contains F but not x.

c. Let X be a metrizable space and $\mathbb{R}$ be the set of real numbers with the standard topology. Let $\mathcal{B}$ be the collection of all sets $B \subseteq X$ such that there is a continuous function $f_B : X \to \mathbb{R}$ and an element $x \in \mathbb{R}$ such that $B = f_B^{-1}(\{x\})$. Show that $\mathcal{B}$ is a basis of closed sets for the metric topology on X.

3. Let $\mathbb{R}_\ell$ be the set of real numbers with the lower-limit topology, that is, the set of half open intervals $[a, b)$ forms a basis for $\mathbb{R}_\ell$. Suppose $A \subseteq \mathbb{R}$ is a subspace of $\mathbb{R}_\ell$ such that for every collection $\mathcal{C}$ of closed sets in A, if each finite subset of $\mathcal{C}$ has non-empty intersection then all the sets in $\mathcal{C}$ have a point in common. Prove that A is closed in $\mathbb{R}_\ell$.

4.

a. Let X be a topological space. Assume for any $p \neq q$ in X there is a continuous function $f : X \to \mathbb{R}$ so that $f(p) \neq f(q)$. Prove that X is Hausdorff.

b. Let Y be an infinite set, with the topology so that $U \subset Y$ is open if and only if U is empty or $X - U$ is finite. Prove that every continuous function $f : Y \to \mathbb{R}$ is a constant.

5.

a. Let X be a topological space. Prove that if X is connected, then for every continuous function $f : X \to \mathbb{R}$, its range $f(X)$ is a point or an interval (open, half open-half closed, closed, with end points possibly infinity).

b. Let Y be a topological space. Assume for every continuous function $f : Y \to \mathbb{R}$, its range $f(Y)$ is a point or an interval (open, half open-half closed, closed, with end points possibly infinity). Prove that Y is connected.

6. Let X be a metric space with distance d, and let $A \subset X$ be a nonempty subset. Define the function

$$f(x) = \inf\{d(x, a) \,|\, a \in A\}.$$

a. Prove that for any $x, y \in X$, $|f(x) - f(y)| \leq d(x, y)$.

b. Prove that f is continuous.

7. Let $\{0, 1\}$ be the set with two elements with discrete topology. Let

$$X = \{0, 1\} \times \{0, 1\} \times \{0, 1\} \times \{0, 1\} \times ...$$

Here there are (countably) infinitely many $\{0, 1\}$'s and X is equiped with the product topology.

Prove that X is separable by finding a countable dense subset in X. (Attention: X is not countable).

8. Let $\mathbb{R}$ be the set of real numbers with the standard topology, and let $f : \mathbb{R} \to Y$ be a surjective map such that if $B \in Y$ is open in Y then f^{-1} is open in $\mathbb{R}$, and if $A \in \mathbb{R}$ is open in $\mathbb{R}$, then $f(A)$ is open in Y. Show Y is second countable.

9. Let $\mathbb{R}_\ell$ be the set of real numbers with the lower-limit topology, that is, the set of half open intervals $[a, b)$ forms a basis for $\mathbb{R}_\ell$. let A be a subspace of $\mathbb{R}_\ell$. Prove or disprove that A is necessarily normal.

10. For each of the following equivalence relations on topological spaces, give the familiar space to which the quotient space $X/\sim$ is homeomorphic. You DO NOT need to justify your answer.

a. Define an equivalence relation on $X = \mathbb{R}^2$ with the standard product topology by setting

$$x_1 \times y_1 \sim x_2 \times y_2 \quad \text{if} \quad y_1 - (x_1)^3 = y_2 - (x_2)^3.$$

b. Define an equivalence relation on $X = \mathbb{R}^2$ with the standard product topology by setting

$$x_1 \times y_1 \sim x_2 \times y_2 \quad \text{if} \quad (x_1)^2 + (y_1)^2 = (x_2)^2 + (y_2)^2.$$

c. Let X be the topological space $\mathbb{R}^2$ with the dictionary order topology. Define an equivalence relation on X by setting

$$x_1 \times y_1 \sim x_2 \times y_2 \quad \text{if} \quad x_1 = x_2.$$

d. Define an equivalence relation on $X = \mathbb{R}$ with the standard topology by setting

$$x \sim y \quad \text{if} \quad x - y \in \mathbb{Q}.$$