Topology Comprehensive Exam

Name:____________________

Answer six (6) questions total. On the first page of your work, please write the numbers of the problems that you want graded. On each page please write only the front side.

(1) Let X be a topological space.

(a) Define what it means for a space X to be *normal*.

(b) Prove that every compact Hausdorff space is normal.

(2)

(a) Prove that if X is first-countable, then for any subset A and $x \in \overline{A}$, there exists a sequence of points $x_n \in A$ such that $\{x_n\}$ converges to x. [Here $\overline{A}$ is the intersection of all closed sets containing A.]

(b) Let $f : X \to Y$, where X and Y are metrizable with metrics d_X and d_Y respectively. Then f is continuous (for every open set in Y, the inverse-image $f^{-1}(Y)$ is open) if and only if for every $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$, such that if $d_X(x, y) < \delta$ then $d_Y(f(x), f(y)) < \epsilon$.

(3) Let X be a topological space

(a) Define what it means for X to be *regular*.

(b) Show that X is regular if and only if given a point x of X and a (open) neighborhood U of x, there is a (open) neighborhood V of x such that $\overline{V} \subseteq U$.

(4) Let $X = \prod_{i=1}^{\infty}[0, 1]$, with the product topology.

(a) Prove that X is Hausdorff.

(b) Prove that X is separable.

(5) Let X be the topological space $\mathbb{R}^2$ with the dictionary order topology.

(a) Prove that X is the same as the product topology on $\mathbb{R}_d \times \mathbb{R}_s$, where $\mathbb{R}_d$ is the discrete topology on $\mathbb{R}$ and $\mathbb{R}_s$ is the standard topology on $\mathbb{R}$.

(b) Define an equivalence relation on X by setting

$$(x_1, y_1) \sim (x_2, y_2) \quad \text{if} \quad y_1 = y_2.$$

Let X^* be the quotient space $X/\sim$. To what familiar space is X^* homeomorphic?

(6)

(a) Prove that if X is path-connected, then X is connected. [You may use principles of topology that do not trivialize the problem.]

(b) Give an example of a space that is connected, but not path connected. [You do not need to prove that the counterexample works.]

(7) Determine if the following are true. If it is true, prove it. If it is false, give a counterexample, and prove that the counterexample works.

(a) A quotient map is is always open.

(b) A bijective quotient map is a homeomorphism.

(c) A surjective open continuous map is a quotient map.

(8)

(a) If X is a topological space, define what it means for a set A to have the subspace topology inherited from X.

(b) If X and Y are topological spaces, define the product topology on $X \times Y$.

(c) If A is a subspace of X and B is a subspace of Y, then the product topology $\mathcal{T}_\times$ on $A \times B$ is the same as the topology $\mathcal{T}_s$ that $A \times B$ inherits as a subspace of $X \times Y$.

(9)

(a) Let X and Y be topological spaces where Y is compact. Let N be an open set containing the set $\{x_0\} \times Y$ of $X \times Y$. Prove that N contains a (open) neighborhood $W \times Y$ of $\{x_0\} \times Y$.

(b) Let X be a topological space. Prove that X is compact if and only if every collection of closed sets $\mathcal{C}$ with the finite intersection property has non-empty intersection, i.e. $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$. [A collection $\mathcal{C}$ of sets has the finite intersection property if for any integer $n > 0$ and any sets $S_1, S_2, \ldots, S_n$ in $\mathcal{C}$, the intersection $S_1 \cap \cdots \cap S_n$ is not empty.]

(10) Let $f : X \to Y$ be a continuous surjective function. Determine which of the following are true. If true prove that it is; and if false, give a counterexample (prove the counterexample works).

(a) If X is connected, then Y is connected.

(b) If X is compact, then Y is compact.

(c) If X is Hausdorff, then Y is Hausdorff.

(d) If X is separable, then Y is separable.