Topology Comprehensive Exam

Name:____________________

Please print out the exam and sign the integrity statement. Allowable resources refer to writing utensils, paper, and electronic devices to be used only for connecting to the exam Zoom session, printing or viewing the exam, scanning written work, and uploading the work for submission. If you do not have a printer, you may handwrite and sign the integrity statement on the first page of your exam. Answer six (6) questions total. On the first page of the exam, please write the numbers for the problems that you want graded.

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Problem							Total
Score							

1. Let $\mathbb{R}^2$ have the standard product topology. Introduce the equivalence relation $\sim$, so that $(x_1, y_1) \sim (x_2, y_2)$ if and only if $(x_1, y_1) = (x_2, y_2)$ or $x_1y_1 = x_2y_2 = 0$ (i.e. all points in the x-axis and y-axis are in a single equivalence class). Prove that the quotient space $\mathbb{R}^2/\sim$ is path-connected.

2. Give an example of a countably infinite subset $A \subset \mathbb{R}$ such that the subspace topology induced on A when $\mathbb{R}$ has the standard topology is different from the subspace topology induced on A when $\mathbb{R}$ has the lower-limit topology. Prove that these two subspace topologies on your set A are distinct.

3. Let, $X = \{1, 2, 3, 4\}$ and $\tau = \{\emptyset, X, \{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$. Note that (X, τ) is a topological space.

a) Show that every two disjoint closed sets have disjoint neighborhoods.

b) Show that (X, τ) is not T_1.

c) Let $A = \{1, 2, 3\} \subset X$ be endowed with the subspace topology. Find disjoint closed subsets of A that do not have disjoint neighborhoods.

4. Let Z be a topological space, X a subset of Z, and let $Y = \overline{X}$ be the closure of X. Determine whether the following are true. If true, prove it is true. If false, give a counterexample, and give a brief explanation for why the counterexample works.

a) If X is connected, then Y is connected.

b) If X is path-connected, then Y is path-connected.

5. Let (X, τ) be a topological space, and let $\mathcal{C}$ be the collection of closed sets. A *filter* on $\mathcal{C}$ is a collection $\mathcal{F}$ of sets from $\mathcal{C}$, such that (1) $\emptyset \notin \mathcal{F}$, (2) if $C_1, C_2 \in \mathcal{F}$, then $C_1 \cap C_2 \in \mathcal{F}$, and (3) if $C_1 \subset C_2$ with $C_1 \in \mathcal{F}$ and $C_2 \in \mathcal{C}$, then $C_2 \in \mathcal{F}$. Show that if each filter on $\mathcal{C}$ has non-empty intersection, then (X, τ) is compact.

6. Let τ be the finite complement topology on $\mathbb{R}$: a set $U \subseteq \mathbb{R}$ is open if and only if U is empty or $\mathbb{R} - U$ is finite.

a) Is $(\mathbb{R}, \tau)$ Hausdorff?

b) Is $(\mathbb{R}, \tau)$ compact?

c) To what values, if any, does the sequence x_n with $x_n = n$ converge in $(\mathbb{R}, \tau)$?

7. For every $x \in [0, 1]$, take a copy of real line, denote it by $\mathbb{R}_x$, and endow it with the standard topology. Consider the product space

$$\mathcal{P} = \prod_{x \in [0,1]} \mathbb{R}_x.$$

If $f \in \mathcal{P}$ and $x \in [0, 1]$, denote by $f(x)$ the $\mathbb{R}_x$-component of f. (Note: if we forget the topology, $\mathcal{P}$ is just the space of all functions from $[0, 1]$ to $\mathbb{R}$).

a) Fix any $x \in [0, 1]$. Prove that the map $\pi : \mathcal{P} \to \mathbb{R}$ defined by $\pi(f) = f(x)$ is continuous.

b) Assume $[0, 1]$ is the standard subspace topology, and $\mathcal{P} \times [0, 1]$ is the product topology. Prove that the map $\Phi : \mathcal{P} \times [0, 1] \to \mathbb{R}$ defined by

$$\Phi(f, x) = f(x)$$

is not continuous.

8. Assume $F_1, F_2, F_3, ...$ is a sequence of closed subsets in a topological space X. Assume for every $x \in X$, there is a neighborhood N_x of x so that $N_x \cap F_j \neq \emptyset$ for only finitely many j. Prove that

$$\bigcup_{j=1}^{\infty} F_j$$

is closed.

9. Let τ be the smallest topology on $\mathbb{R}^2$ such that the intersection of any two lines is open.

a) Is $(\mathbb{R}^2, \tau)$ 1st countable?

b) Is $(\mathbb{R}^2, \tau)$ 2nd countable?

c) Is $(\mathbb{R}^2, \tau)$ metrizable?

10. Let (X, τ) be a topological space. For any subset B of X, let $\overline{B}$ denote the closure of B in X and let B° the interior of B in X. Let A a subset of X. Determine whether the following are true or false. If true, prove it is so, and if false, give a counterexample.

a) The boundary of A (that is, $\overline{A} - A^\circ$) contains all the limit points of A

b) $\overline{(A^\circ)}$ contains all the limit points of A.

c) The set of limit points of A contains all limit points of $\overline{A}$.

d) If $x_1, x_2, \ldots$ is a sequence of points in A with limit $x \in X$, then x is a limit point of A.