California State University, Long Beach
Department of Mathematics and Statistics

Real Analysis Comprehensive Examination

Spring 2024

Choose **six** of the nine problems. On the first page of your work, please write the numbers of the problems you would like graded.

Notation: For a measurable subset $A \subseteq \mathbb{R}^d$, let $\lambda(A)$ denote the Lebesgue measure of A.

1. Definition: Let $|I|$ denote the length of the nonempty open interval I. The outer measure of a set $E \subseteq \mathbb{R}$, denoted $m^*(E)$, is defined to be

   $$m^*(E) = \inf\{\sum_{k=1}^{\infty} |I_k| : I_1, I_2, \ldots \text{ is a sequence of open intervals with } E \subseteq \bigcup_{k=1}^{\infty} I_k\}.$$

   a. Prove that m^* is subadditive, directly from the definition. In other words, prove that if $A, B \subseteq \mathbb{R}$, then $m^*(A \cup B) \leq m^*(A) + m^*(B)$.
   b. Prove that m^* is monotonic, directly from the definition. In other words, prove that if $A, B \subseteq \mathbb{R}$ with $A \subseteq B$, then $m^*(A) \leq m^*(B)$.

2. a. State Egorov's Theorem.
   b. Let $\{f\}_n$ be a uniformly bounded sequence of Lebesgue measurable functions defined on a measurable set E of finite measure. Use Egorov's theorem to prove that if $f_n \to f$ a.e., then

   $$\int_E f_n d\lambda \to \int_E f d\lambda.$$

3. a. State the Monotone Convergence Theorem.
   b. Give an example to show that the following statement is FALSE:
   Let $f_1 \geq f_2 \geq \cdots$ be a decreasing sequence of non-negative measurable real-valued functions defined on $\mathbb{R}$. Let $f : \mathbb{R} \to [-\infty, \infty]$ be the limit function $f(x) = \lim_{k\to\infty} f_k(x)$. Then

   $$\lim_{k\to\infty} \int_{\mathbb{R}} f_k \, d\lambda = \int_{\mathbb{R}} f \, d\lambda.$$

   (That is, if the increasing sequence in the Monotone Convergence Theorem were replaced by a decreasing sequence, the statement would be false.)

   c. Give an example to show that the following statement is FALSE:
   Let $f_1 \geq f_2 \geq \cdots$ be an increasing sequence of measurable functions defined on $\mathbb{R}$. Let $f : \mathbb{R} \to [-\infty, \infty]$ be the limit function $f(x) = \lim_{k\to\infty} f_k(x)$. Then

   $$\lim_{k\to\infty} \int_{\mathbb{R}} f_k \, d\lambda = \int_{\mathbb{R}} f \, d\lambda.$$

   (That is, if the hypothesis that the functions in the sequence are non-negative were dropped from the Monotone Convergence Theorem, the statement would be false.)

4. Assume $f\colon \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable and that
$$\int_{\mathbb{R}} |f|^2 d\lambda \le 1.$$
Prove that for any $n > 0$, the Lebesgue measure of $\{x \in \mathbb{R}: f(x) \ge n\}$ is at most $\frac{1}{n^2}$.

5. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a function. For $k \in \mathbb{Z}$, let
$$G_k := \left\{ x \in \mathbb{R} : \exists \delta > 0 \text{ such that } |f(b) - f(c)| < \frac{1}{k} \text{ for all } b, c \in (x - \delta, x + \delta) \right\}$$
   a. Prove that G_k is an open subset of $\mathbb{R}$ for each $k \in \mathbb{Z}^+$.
   b. Prove that the set of points at which f is continuous equals $\bigcap_{k=1}^{\infty} G_k$.
   c. Conclude that the set of points at which f is continuous is a Borel set.

6. Let $\mathcal{T}$ be the smallest σ-algebra that contains the collection $\{(-\infty, r]: r \in \mathbb{Q}\}$. Prove that $\mathcal{T}$ is equal to the Borel σ-algebra.

7. Let $F_k \subset [0,1], k \in \mathbb{N}$ be measureable sets, and assume there exists $\delta > 0$ such that $\lambda(F_k) \ge \delta$ for all k. Also assume there is a nonnegative sequence $a_k \ge 0$ which satisfies:
$$\sum_{k=1}^{\infty} a_k \chi_{F_k(x)} < \infty \text{ for } x \in [0,1].$$
Show that
$$\sum_{k=1}^{\infty} a_k < \infty.$$

8. Suppose that a measurable set $E \subset (0,1)$ is such that $\lambda\big(E \cap (r,s)\big) \ge \frac{s-r}{4}$ for all rational $0 < \mathrm{r} < \mathrm{s} < 1$. Prove $\lambda(E) \ge 1/4$.

9. True or False: If A is measure 0, then A is countable. If the statement is true, prove it. If the statement is false, construct a counterexample.