California State University, Long Beach
Department of Mathematics and Statistics

Real Analysis Comprehensive Examination

Sept. 2024

Choose six of the nine problems, and circle the numbers of the problems that you want graded.

Notation: For a Lebesgue measurable subset $A \subseteq \mathbb{R}$, let $\lambda(A)$ denote the Lebesgue measure of A. For any subset $C \subseteq \mathbb{R}$, we let $m_*(C)$ be the outer measure of C.
For any set D, let χ_D denote the characteristic function on D, i.e. $\chi_D(x) = 1$ if $x \in D$ and $\chi_D(x) = 0$ otherwise.

1. Prove directly from the definition of outer measure that the Lebesgue outer measure of the unit interval $[0, 1]$ is one. That is, prove $m_*([0, 1]) = 1$

2. Let X be a set.
   (a) Define what it means for $\mathcal{S}$ to be a σ-algebra on X.
   (b) Let $\mathcal{B}$ be the Borel σ-algebra on $\mathbb{R}$. Show that $\mathcal{B}$ is closed under constant multiples, i.e. if $r \in \mathbb{R}$ and $B \in \mathcal{B}$, then $rB \in \mathcal{B}$.

3. True or False? Let $(X, \mathcal{S}, \mu)$ be a measure space and $E_1 \supseteq E_2 \supseteq \cdots$ be a decreasing sequence of sets in $\mathcal{S}$. Then

$$\mu\left(\bigcap_{k=1}^{\infty} E_k\right) = \lim_{k\to\infty} \mu(E_k).$$

   If it is true, prove it. If it is false, give a counterexample.

4. Suppose $g : \mathbb{R} \to [0, \infty]$ is Lebesgue measurable, and that $\int g \, \mathrm{d}\lambda < \infty$. Prove that for all $\epsilon > 0$, there exists $\delta > 0$ such that if B is a measurable set with $\lambda(B) < \delta$, then

$$\int_B g \, \mathrm{d}\lambda < \epsilon$$

5. Let $E \subseteq \mathbb{R}$ be a Lebesgue measurable set. Recall that a sequence $f_n : E \to \mathbb{R}$ is said to *converge in measure* to a function $f : E \to \mathbb{R}$ provided that $\lim_{n\to\infty} \lambda(\{x \in E : \ |f_n(x) - f(x)| \geq 0\}) = 0$.

   Consider the sequence of intervals E_n such that

$$E_1 = \left[0, \frac{1}{2}\right), E_2 = \left[\frac{1}{2}, 1\right],$$

$$E_3 = \left[0, \frac{1}{3}\right), E_4 = \left[\frac{1}{3}, \frac{2}{3}\right), E_5 = \left[\frac{2}{3}, 1\right],$$

$$E_6 = \left[0, \frac{1}{4}\right), E_7 = \left[\frac{1}{4}, \frac{1}{2}\right), \ldots$$

   and so on. Let $f_n : [0, 1] \to \mathbb{R}$ be given by $f_n(x) = \begin{cases} 0 & \text{if } x \in E_n \\ 1 & \text{otherwise} \end{cases}$.

   (a) Prove that f_n converges in measure to $\chi_{[0,1]}$.
   (b) Prove that f_n does not converge to $\chi_{[0,1]}$ pointwise almost everywhere.

6. (a) State the Monotone Convergence Theorem.
   (b) Prove the Monotone Convergence Theorem.

7. Fatou's lemma is the following theorem: Let $(X, \mathcal{S}, \mu)$ be a measure space and $f_1, f_2, \ldots$ be a sequence of nonnegative $\mathcal{S}$-measurable functions on X. Define a function $f : X \to [0, \infty]$ by $f(x) = \liminf_{k\to\infty} f_k(x)$. Then f is $\mathcal{S}$-measurable and $\displaystyle\int f \, \mathrm{d}\mu \leq \liminf_{k\to\infty} \int f_k \, \mathrm{d}\mu$. Below we will use Fatou's lemma to give a proof of a version of the bounded convergence theorem.

   Let $(X, \mathcal{S}, \mu)$ be a measure space with $\mu(X) < \infty$. Let $f_k : X \to \mathbb{R}$ be a sequence of $\mathcal{S}$-measurable functions such that $f_k \to f$ pointwise to some function $f : X \to \mathbb{R}$. Suppose there exists a $C \in (0, \infty)$ such that $\sup_{k\in\mathbb{N}, x\in X} |f_k(x)| \leq C$.

   (a) Define the function $F_k = C - f_k$. Explain why we can apply Fatou's lemma and show that

   $$-\int f \, \mathrm{d}\mu \leq \liminf_{k\to\infty} \int -f_k \, \mathrm{d}\mu$$

   Then explain why we may conclude $\displaystyle\limsup_{k\to\infty} \int f_k \, \mathrm{d}\mu \leq \int f \, \mathrm{d}\mu$.

   (b) Define the function $F_k' = C + f_k$, and use it to show that

   $$\int f \, \mathrm{d}\mu \leq \liminf_{k\to\infty} \int f_k \, \mathrm{d}\mu$$

   Then explain why we may conclude

   $$\int f \, \mathrm{d}\mu = \lim_{k\to\infty} \int f_k \, \mathrm{d}\mu$$

8. Guess what the following limit is:

   $$\lim_{n\to\infty} \int_{[6,n]} \sum_{k=0}^{n} \frac{x^k e^{-2x}}{k!} \, \mathrm{d}x$$

   Then prove your conjecture.

9. Let $f : [0, 1] \to \mathbb{R}$ be given by

   $$f(x) = \begin{cases} 3 - x^2 & \text{if } x \text{ is irrational} \\ 3/x^2 & \text{if } x \neq 0 \text{ and } x \text{ is rational} \\ 42 & \text{if } x = 0 \end{cases}$$

   (a) For what $x \in [0, 1]$ is f continuous?

   (b) Is f Lebesgue integrable? If so, what is its integral on $[0, 1]$?