California State University, Long Beach
Department of Mathematics and Statistics

Real Analysis Comprehensive Examination

Fall 2023

Choose **six** of the nine problems. On the first page of your work, please write the numbers of the problems you would like graded.

Notation:

- **For a measurable subset $A \subseteq \mathbb{R}^d$, let $\lambda(A)$ denote the Lebesgue measure of A.**
- **For a function $f: E \to [-\infty, \infty]$, we say f is Lebesgue integrable on E provided $\int_E |f| < \infty$.**

1. a. State the Monotone Convergence Theorem.
   b. Use the Monotone Convergence Theorem to show that the function
   $$f(x) = x^{-0.5}\chi_{(0,1)}(x)$$
   is integrable, and evaluate $\int_{-\infty}^{\infty} f\, d\lambda$. Hint: Consider $f_n(x) = x^{-0.5}\chi_{\left[\frac{1}{n},1\right]}(x)$.
   c. Show that $f(x) = x^{-0.5}\chi_{(1,\infty)}(x)$ is not integrable.

2. Let $E \subseteq \mathbb{R}$ be measurable, and let $f: E \to [0, \infty)$ be integrable. Prove that
   $$\lambda(\{x \in E | f(x) > \alpha\}) \leq \frac{1}{\alpha}\int_E f\, d\lambda.$$

3. a. State Egorov's Theorem.
   b. Let $\{f\}_n$ be a bounded sequence of Lebesgue measurable functions defined on a measurable set E of finite measure. Use Egorov's theorem to prove that if $f_n \to f$ a.e., then
   $$\int_E f_n d\lambda \to \int_E f d\lambda.$$

4. Let $\mathcal{B}$ denote the σ-algebra of Borel sets.
   a. Let $\mathcal{H}$ be the smallest σ-algebra containing all intervals of the form $[a, \infty)$, where $a \in \mathbb{R}$. Prove that $\mathcal{H}$ is equal to the σ-algebra of Borel sets $\mathcal{B}$.
   b. A function $f: \mathbb{R} \to \mathbb{R}$ is called *Borel measurable* if for all open sets $U \subset \mathbb{R}$, we have $f^{-1}(U) \in \mathcal{B}$. Prove that $f: \mathbb{R} \to \mathbb{R}$ is Borel measurable if and only if for all $a \in \mathbb{R}$, we have $f^{-1}([a, \infty)) \in \mathcal{B}$.

5. Let $E \subseteq \mathbb{R}$ be measurable, and let $f: E \to [0, \infty)$. Suppose $\int_E f\, d\lambda = 0$. Prove that $f(x) = 0$ for $a.e. x \in E$.

6. Assume f is Lebesgue integrable on $\mathbb{R}$. Define
$$f_n(x) = \begin{cases} f(x) & \text{if} \quad |f(x)| > n \\ 0 & \text{if} \quad |f(x)| \leq n \end{cases}$$
Prove that
$$\lim_{n\to\infty} \int_{\mathbb{R}} |f_n| d\lambda = 0$$

7. Assume that $A_1, A_2, \ldots, A_n$ are Lebesgue measurable subsets in [0,1]. Assume
$$\sum_{j=1}^{n} \lambda(A_j) > n - 1.$$
Prove that $\lambda\left(\bigcap_{j=1}^{n} A_j\right) > 0$.

8. Suppose $g: \mathbb{R} \to [0, \infty]$ is Lebesgue measurable, and that $\int g \, d\lambda < \infty$. Prove that for all $\varepsilon > 0$, there exists $\delta > 0$ such that if B is a measurable set with $\lambda(B) < \delta$, then
$$\int_B g \, d\mu < \varepsilon.$$

9. Assume $f: \mathbb{R} \to \mathbb{R}$ is Lebesgue integrable on $[0, \infty)$. Suppose $f(0) = 0$, and $f'(0)$ exists. Prove that $g(x) = \frac{f(x)}{x}$ is Lebesgue integrable on $[0, \infty)$.