California State University, Long Beach
Department of Mathematics and Statistics

Real Analysis Comprehensive Examination

February 20, 2021

Choose six of the nine problems. On the first page of your work, please write the numbers of the problems that you want graded.

Notations: for A a measurable subset of $\mathbb{R}^n$, $\lambda(A)$ is the Lebesgue measure of A. The function $\chi_A(x)$ is the function that equals 1 for $x \in A$ and equals zero for $x \notin A$.

1. (a) Let $\mathcal{A}$ be the σ-algebra generated by all sets in $\mathbb{R}$ of the form $[a, \infty)$. Let $\mathcal{B}$ be the σ-algebra of Borel sets in $\mathbb{R}$. Prove $\mathcal{A} = \mathcal{B}$.

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Assume for all $a \in \mathbb{R}$, $f^{-1}([a, \infty))$ is a Borel set. Prove that for any Borel set B, $f^{-1}(B)$ is a Borel set.

2. Assume $f \in L^1(0, \infty)$. Define

$$F(x) = \int_0^\infty e^{-xt} f(t)dt.$$

   Prove that F is uniformly continuous for $0 \leq x < \infty$.

3. Prove that for all Lebesgue integrable $f, g : \mathbb{R}^n \to \mathbb{R}$ the convolution $f * g$,

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t - x)g(x)\, dx$$

   is Lebesgue integrable.

4. Convergence of the sequence f_n to f in measure on (0,1) means that

$$\text{for all } \varepsilon > 0, \lim_{n\to\infty} \lambda\{x \in (0, 1) |\, |f_n(x) - f(x)| \geq \varepsilon\} = 0.$$

   Show that convergence in measure of f_n to f on the interval (0,1) is implied by pointwise convergence almost everywhere on (0,1), that is, by $f_n(x) \to f(x)$ for almost every $x \in (0, 1)$.

5. Let f be a Lebesgue integrable function on $\mathbb{R}$. Prove that for any $L > 0$

$$\lim_{n\to\infty} \int_n^{n+L} f(x)dx = 0.$$

6. (a) Suppose f is an integrable function on E and suppose that $f \geq 0$ on E. For $\alpha > 0$, define $E_\alpha = \{x \in E \mid f(x) > \alpha\}$. Prove that

$$\mu(E_\alpha) \leq \frac{1}{\alpha} \int_E f.$$

   (b) Suppose that $f \geq 0$ on E and $\displaystyle\int_E f = 0$. Prove that $f(x) = 0$ almost everywhere on E.

7. (a) Show that the set of rational numbers in $[0, 1]$ is of measure 0.

   (b) Show there is a closed subset $A \subset [0, 1]$ that contains no rational number such that $\lambda(A) > 0$.

8. Let $g(x) = \dfrac{1}{x^{1/3}} \chi_{(0,1)}(x)$, and let r_n be an enumeration of the rational numbers in $\mathbb{R}$. Let $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} g(x - r_n)$.

   Show that $f \in L^1(\mathbb{R})$ and that $f(x)$ is finite almost everywhere.

9. Suppose that $0 < p < q < r < \infty$, and suppose that $\displaystyle\int_{\mathbb{R}} |f|^p < \infty$ and $\displaystyle\int_{\mathbb{R}} |f|^r < \infty$. Prove that $\displaystyle\int_{\mathbb{R}} |f|^q < \infty$.