California State University, Long Beach
Department of Mathematics and Statistics

Real Analysis Comprehensive Examination

February 15, 2020

Choose six of the nine problems. On the first page of your work, please write the numbers of the problems that you want graded.

Definition: Let $|I|$ denote the length of the closed interval I. The exterior measure of E, $m_*(E)$, $E \subset \mathbb{R}$, is defined to be $\inf\{\sum |I_j|\}$ where the infimum is taken over all countable coverings of E by closed intervals, that is, $E \subset \cup_j I_j$.

Notation: $m(A)$ denotes the Lebesgue measure of a measurable set A.

1. Prove directly from the definition of exterior measure that the exterior measure of the interval $[0,1]$ is one. That is, prove that $m_*([0,1]) = 1$.

2. Suppose E is a set in $\mathbb{R}$. Let $E_n = E \cap [-n, n]$, $n \in \mathbb{Z}^+$. Prove or disprove: If $m(E_n) < m(E_{n+1})$ for all $n \in \mathbb{Z}^+$, then $m(E) = \infty$.

3. Suppose E is a measurable set in $\mathbb{R}$. Suppse $f(x)$ is a continuous real–valued function on $\mathbb{R}$ satisfying $|f(y) - f(x)| \leq 2|y - x|$ for all real numbers x and y. Let $f(E) = \{f(x) : x \in E\}$. Prove or disprove: if $m(E) = 0$ then $m(f(E)) = 0$.

4. Let $E = [0,1] - \mathbb{Q}$, that is, E is the set of irrational numbers between 0 and 1. Given $\epsilon > 0$, construct a closed set F contained in E such that $m_*(E - F) < \epsilon$.

5. Let K be a compact set in $\mathbb{R}^d$, and let $O_n = \{x : d(x, K) < 1/n\}$, where $d(x, K) = \inf\{|x - y| : y \in K\}$. Prove that $\lim_{n\to\infty} m(O_n) = m(K)$. Show by an example that this need not hold if K is assumed to be a closed set instead of compact.

6. Let $C[-1,1]$ be the vector space of continuous functions in the interval $[-1,1]$. Show that this vector space is not complete in the norm given by
$$\|f\| = \int_{-1}^{1} |f(x)|\, dx$$

7. Construct a sequence of integrable functions which converges in $L^1(\mathbb{R})$ but which does not converge almost everywhere.

8. State and prove the Dominated Convergence Theorem. In the proof you may use any other results in integration theory, such as Fatou's Lemma and the Bounded Convergence Theorem.

9. The point of this problem is to show that $\int_E f$ can be defined in the case that $m(E) < \infty$ for a bounded, measurable function f, assuming that the integral for simple functions has already been defined.

   Let E be a measurable set in $\mathbb{R}^d$ with $m(E) < \infty$. Suppose that $f_n(x)$, $n \in \mathbb{Z}^+$, are simple functions on E and that $f_n(x) \to f(x)$ for all $x \in E$. Suppose further that for all $x \in E$, and for all $n \in \mathbb{Z}^+$, $|f_n(x)| \leq M$ for some number M. Prove that $\lim_{n\to\infty} \int_E f_n$ is a Cauchy sequence, hence convergent, by applying Egorov's theorem.