California State University, Long Beach
Department of Mathematics and Statistics

# Real Analysis Comprehensive Examination

September 14, 2019

**Choose six of the nine problems. On the first page of your work, please write the numbers of the problems that you want graded.**

**Definition:** Let $|I|$ denote the length of the closed interval $I$. The exterior measure of $E$, $m_*(E)$, $E \subset \mathbb{R}$, is defined to be $\inf\{\sum |I_j|\}$ where the infimum is taken over all countable coverings of $E$ by closed intervals, that is, $E \subset \cup_j I_j$.

1. Prove directly from the definition of exterior measure that the exterior measure of the unit interval $[0,1]$ is one. That is, prove that $m_*([0,1]) = 1$.

2. (a) Prove the subadditivity property of Lebesgue exterior measure directly from its definition (see above for the definition). That is, prove that
$$m_*(E_1 \cup E_2) \leq m_*(E_1) + m_*(E_2),$$
where $E_1$ and $E_2$ are arbitrary subsets of $\mathbb{R}$.
   (b) Prove that if $A \subset B$ then $m_*(A) \leq m_*(B)$, where $A$ and $B$ are any subsets of $\mathbb{R}$.

3. Show that for any measurable set $E$ in $\mathbb{R}$ with $m(E) = 1$ there is a mesurable set $A$, $A \subset E$ with $m(A) = 1/2$.

4. Prove or disprove the following assertion: If $f(x)$ is a measurable function on the interval $[0,1]$ taking values in $\mathbb{R}$, then $\lim_{N\to\infty} m(E_N) = 0$, where $E_N = \{x \in [0,1] : f(x) \geq N\}$, $N \in \mathbb{Z}^+$.

5. Let $f$ be a measurable function on $\mathbb{R}^d$ taking values in $[0, M]$, where $M$ is a positive real number. Show that there exists a sequence of simple functions $f_n$ taking values in $[0, M]$ that converge uniformly to $f$.

6. Guess what the following limit is:
$$\lim_{n\to\infty} \int_6^n \left(1 + \frac{x}{n}\right)^n \exp(-2x)\, dx$$
Prove your conjecture. The integral is understood to be a Lebesgue integral.

7. (a) Prove or disprove: Let $E = \cup_{n=1}^{\infty}[n, n + n^{-2}]$. If $f_n(x)$ are non–negative integrable functions on $E$ converging uniformly to $f$ on $E$, then
$$\lim_{n\to\infty} \int_E f_n = \int_E f$$
   (b) Prove or disprove: If $f_n(x)$ are non–negative integrable functions on $[0,\infty)$ converging uniformly to $f$ on $[0,\infty)$, then
$$\lim_{n\to\infty} \int_0^{\infty} f_n = \int_0^{\infty} f$$

8. Let $f_h(x) = f(x-h)$, where $x$ and $h$ are in $\mathbb{R}^d$. Show that if $f : \mathbb{R}^d \to \mathbb{R}$ is continuous and has bounded support, then $f_h \to f$ in $L^1$ as $h \to 0$. (Hint: Use uniform continuity.) Then prove that $f_h \to f$ in $L^1$ as $h \to 0$ when $f$ is in $L^1$.

9. Use Egorov's Theorem to prove the Bounded Convergence Theorem given below:

   Let $E$ be a measurable set in $\mathbb{R}^d$ with $m(E) < \infty$. Suppose that $f_n(x)$, $n \in \mathbb{Z}^+$, are measurable functions on $E$ satisfying $|f_n(x)| \leq M$ for some number $M$, for all $x \in E$, and all $n \in \mathbb{Z}^+$. If $f_n(x) \to f(x)$ for all $x \in E$, then $\lim_{n\to\infty} \int_E f_n = \int_E f$.