Formulas, PDE comprehensive exam

- The **characteristic equations** for the non–linear **first order** equation $F(x, y, z, p, q) = 0$, $z = u$, $p = u_x$, $q = u_y$, are given by

$$dx/dt = F_p \qquad dy/dt = F_q \qquad dz/dt = pF_p + qF_q \qquad dp/dt = -F_x - F_z p \qquad dq/dt = -F_y - F_z q$$

- **Green's identities:**

$$\begin{aligned} \int_\Omega (g\Delta f - f\Delta g)\, dx &= \int_{\partial\Omega} (g\partial_n f - f\partial_n g)\, dS \\ \int_\Omega (g\Delta f + \nabla g \nabla f)\, dx &= \int_{\partial\Omega} g\partial_n f\, dS \\ \int_\Omega \Delta f\, dx &= \int_{\partial\Omega} \partial_n f\, dS \end{aligned}$$

where ∂_n is the (outward) normal derivative.

- The **fundamental solution of the Laplace operator** Δ in $\mathbb{R}^n$ is given by the potential

$$K(x) = \begin{cases} (2\pi)^{-1} \log \|x\| & \text{if } n = 2 \\ -(4\pi\|x\|)^{-1} & \text{if } n = 3 \end{cases}$$

- The **Poisson integral formula** is $u(\xi) = \int_{\partial\Omega} H(x, \xi)u(x)dS_x$, where $H(x, \xi)$ is the **Poisson kernel**. The Poisson kernel in the upper half-space in $\mathbb{R}^n$ (that is, $\xi_n > 0$) is

$$H(x', \xi) = \frac{2\xi_n}{\omega_n |x' - \xi|^n} \qquad x' = (x_1, \ldots, x_{n-1})$$

The Poisson kernel for the unit ball in $\mathbb{R}^n$ is

$$H(x, \xi) = \frac{1 - |\xi|^2}{\omega_n |x - \xi|^n} \qquad \|x\| = 1$$

- **Kirchoff's formula** gives the solution to the pure initial value problem for the three dimensional **wave equation** $u_{tt} = c^2\Delta u$ with initial data $u(x, 0) = g(x)$, $u_t(x, 0) = h(x)$.

$$u(x, t) = (4\pi)^{-1} \frac{\partial}{\partial t} \left(t \int_{\|\xi\|=1} g(x + ct\xi)\, dS_\xi \right) + (4\pi)^{-1} t \int_{\|\xi\|=1} h(x + ct\xi)\, dS_\xi$$

- The solution to the pure initial value problem for the **heat equation** $u_t = \Delta u$ with initial condition $u(x, 0) = g(x)$ is given by the convolution $u(x, t) = \int_{\mathbb{R}^n} K(x - y, t)g(y)\, dy$ of the heat kernel $K(x, t)$ with the initial data. The heat kernel for $n = 1$ is given by

$$K(x, t) = (4\pi t)^{-1/2} \exp(-x^2/4t)$$

- The **Fourier transform** $\mathcal{F}g$ and the inverse Fourier transform $\mathcal{F}^{-1}h$ are

$$\mathcal{F}g(\xi) = \int_{\mathbb{R}^n} \exp(-ix \cdot \xi)g(x)\, dx, \qquad \mathcal{F}^{-1}h(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \exp(-ix \cdot \xi)h(\xi)\, d\xi$$

Fourier inversion formula: $\mathcal{F}^{-1}(\mathcal{F}g) = g$. Basic formula: $\mathcal{F}(\partial_k g)(\xi) = i\xi_k \mathcal{F}g(\xi)$.