Formulas, PDE comprehensive exam

- The **characteristic equations** for the non–linear **first order** equation $F(x, y, z, p, q) = 0$, $z = u$, $p = u_x$, $q = u_y$, are given by

  $$dx/dt = F_p \qquad dy/dt = F_q \qquad dz/dt = pF_p + qF_q \qquad dp/dt = -F_x - F_z p \qquad dq/dt = -F_y - F_z q$$

- **Green's identities:**

  $$\begin{aligned} \int_\Omega (g\Delta f - f\Delta g)\,dx &= \int_{\partial\Omega} (g\partial_n f - f\partial_n g)\,dS \\ \int_\Omega (g\Delta f + \nabla g \nabla f)\,dx &= \int_{\partial\Omega} g\partial_n f\,dS \\ \int_\Omega \Delta f\,dx &= \int_{\partial\Omega} \partial_n f\,dS \end{aligned}$$

  where ∂_n is the (outward) normal derivative.

- The **fundamental solution of the Laplace operator** Δ in $\mathbb{R}^n$ is given by the potential

  $$K(x) = \begin{cases} (2\pi)^{-1}\log\|x\| & \text{if } n = 2 \\ -(4\pi\|x\|)^{-1} & \text{if } n = 3 \end{cases}$$

- The **Poisson integral formula** is $u(\xi) = \int_{\partial\Omega} H(x, \xi)u(x)dS_x$, where $H(x, \xi)$ is the **Poisson kernel**. The Poisson kernel in the upper half-space in $\mathbb{R}^n$ (that is, $\xi_n > 0$) is

  $$H(x', \xi) = \frac{2\xi_n}{\omega_n |x' - \xi|^n} \qquad\qquad x' = (x_1, \ldots, x_{n-1})$$

  The Poisson kernel for the unit ball in $\mathbb{R}^n$ is

  $$H(x, \xi) = \frac{1 - |\xi|^2}{\omega_n |x - \xi|^n} \qquad\qquad \|x\| = 1$$

- **Kirchoff's formula** gives the solution to the pure initial value problem for the three dimensional **wave equation** $u_{tt} = c^2\Delta u$ with initial data $u(x, 0) = g(x)$, $u_t(x, 0) = h(x)$.

  $$u(x, t) = (4\pi)^{-1}\frac{\partial}{\partial t}\left(t\int_{\|\xi\|=1} g(x + ct\xi)\,dS_\xi\right) + (4\pi)^{-1}t\int_{\|\xi\|=1} h(x + ct\xi)\,dS_\xi$$

- The solution to the pure initial value problem for the **heat equation** $u_t = \Delta u$ with initial condition $u(x, 0) = g(x)$ is given by the convolution $u(x, t) = \int_{\mathbb{R}^n} K(x - y, t)g(y)\,dy$ of the heat kernel $K(x, t)$ with the initial data. The heat kernel for $n = 1$ is given by

  $$K(x, t) = (4\pi t)^{-1/2}\exp(-x^2/4t)$$

- The **Fourier transform** $\mathcal{F}g$ and the inverse Fourier transform $\mathcal{F}^{-1}h$ are

  $$\mathcal{F}g(\xi) = \int_{\mathbb{R}^n} \exp(-ix\cdot\xi)g(x)\,dx, \qquad \mathcal{F}^{-1}h(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} \exp(-ix\cdot\xi)h(\xi)\,d\xi$$

  Fourier inversion formula: $\mathcal{F}^{-1}(\mathcal{F}g) = g$. Basic formula: $\mathcal{F}(\partial_k g)(\xi) = i\xi_k\mathcal{F}g(\xi)$.

Spring 2022

Partial Differential Equation Comprehensive Exam

Please read and sign the integrity statement below, and attach it to the answer that you submit to dropbox.

I will not share the contents of this comprehensive exam with any person or site. I have only used allowable resources for this comprehensive exam. I have neither given nor received help during this comprehensive exam.

Name _______________ Signature _______________

Do any six problems. Clearly indicate in the table below which problems you want to be graded. If you do not select any problems we will grade the first 6 problems. Good luck!

Problems	1	2	3	4	5	6	7	8

1. Solve the following initial value problem using characteristics.

$$u_x^2 + u_y^2 = u$$

with the initial condition $u(0, y) = ay^2$. For what positive a are there solutions? Is the solution unique?

2. Consider the second order linear equation $x^2 u_{xx} - y^2 u_{yy} = 0$

   (a) Classify the equation as hyperbolic, parabolic, or elliptic.

   (b) Rewrite this equation in its canonical form.

3. Let $\Omega \subset \mathbb{R}^n$ denote a bounded, connected domain with smooth boundary. Use Green's identity and the energy method to show that $u(\mathbf{x}, t) = 0$ is the unique solution to the following parabolic PDE with bi-harmonic diffusion:

$$\begin{aligned} u_t &= -\Delta(\Delta u) && \mathbf{x} \in \Omega,\ t > 0, \\ \Delta u(\mathbf{x}, t) &= 0 && \mathbf{x} \in \partial\Omega,\ t > 0, \\ u(\mathbf{x}, t) &= 0 && \mathbf{x} \in \partial\Omega,\ t > 0, \\ u(\mathbf{x}, 0) &= 0 && \mathbf{x} \in \Omega,\ t = 0. \end{aligned}$$

4. (a) Green's identity is given by

$$\int_\Omega (g\Delta f - f\Delta g)\mathrm{d}x = \int_{\partial\Omega} (g\partial_n f - f\partial_n g)$$

   where ∂_n is the normal derivative. Prove this by applying the divergence theorem.

   (b) Let $K(x)$ denote the fundamental solution of the Laplace operator Δ in $\mathbb{R}^3$, and let $v(x)$ be an infinitely differentiable function which equals zero for $|x| > R$. Apply Green's identity to prove the following identity:

$$\int_{\mathbb{R}^3} K(x)v(x)\mathrm{d}x = v(0)$$

5. If Ω is a bounded open set in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. Show that if u satisfies

$$\Delta u = 0 \quad \text{in} \quad \Omega$$

then, using the mean value property for harmonic functions to show

$$\max_\Omega u = \max_{\partial\Omega} u$$

6. (a) Verify that $u(x,t) = F(x+ct) + G(x-ct)$, F and G twice differentiable, is a solution of the wave equation
$$u_{tt} = c^2 u_{xx}$$
Use this to solve the initial value problem for the wave equation with initial conditions
$$\begin{aligned} u(x,0) &= f(x), \quad \text{for} \quad x \in \mathbb{R}, \\ u_t(x,0) &= g(x), \quad \text{for} \quad x \in \mathbb{R}. \end{aligned}$$
Verify your solution.

   (b) Solve the initial boundary problem for the wave equation on the quarter plane $\{(x,t) : x > 0, t > 0\}$ with general initial conditions, as above, but for $x > 0$, and boundary condition $u(0,t) = 0$, for $t > 0$.

7. Consider the wave equation in the first quadrant $x > 0, t > 0$
$$\begin{aligned} u_{tt} &= u_{xx}, && 0 < x < \infty,\ t > 0, \\ u(x,0) &= f(x), && 0 < x < \infty, \\ u_t(x,0) &= g(x), && 0 < x < \infty, \\ u(0,t) &= 0, && t > 0, \end{aligned}$$
where $f \in C^2([0,\infty))$ and $g \in C^1([0,\infty))$ satisfy $f(0) = f'(0) = g(0) = 0$.

   (a) Solve the problem using the odd extensions of f and g.
   (b) Sketch the domain of dependence of a point (x_0, t_0) where $0 < x_0 < \infty$ and $t_0 > 0$.
   (c) Sketch the region of influence of a point x_0 where $0 < x_0 < \infty$.

8. Let $\Omega = B_1(\mathbf{0})$ denote the unit ball in $\mathbb{R}^2$ centered at the origin. Show the solution to
$$\begin{aligned} u_t(\mathbf{x},t) &= \Delta u(\mathbf{x},t) && \text{in} \quad \Omega_T := \{(\mathbf{x},t) : \mathbf{x} \in \Omega,\ 0 < t < T\} \\ u(\mathbf{x},t) &= h(\mathbf{x},t) && \mathbf{x} \in \partial\Omega,\ t > 0 \\ u(\mathbf{x},0) &= g(\mathbf{x}) && \mathbf{x} \in \Omega,\ t = 0 \end{aligned} \tag{1}$$
satisfies the inequality
$$\mathrm{e}^{-8t}\left(1 - |\mathbf{x}|^2\right)^2 \le u(\mathbf{x},t) \le \mathrm{e}^{-4t}(1 - |\mathbf{x}|^2)$$
if $g(\mathbf{x}) = 1 - |\mathbf{x}|^2$ and $h(\mathbf{x},t) = 0$. You may use the identities $\Delta|\mathbf{x}|^2 = 4$ and $\Delta|\mathbf{x}|^4 = 16|\mathbf{x}|^2$ (valid in two dimensions) without proof.