Fall 2020

Partial Differential Equation Comprehensive Exam

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Formulas, PDE comprehensive exam

- The **characteristic equations** for the non–linear **first order** equation $F(x, y, z, p, q) = 0$, $z = u$, $p = u_x$, $q = u_y$, are given by

$$dx/dt = F_p \qquad dy/dt = F_q \qquad dz/dt = pF_p + qF_q \qquad dp/dt = -F_x - F_z p \qquad dq/dt = -F_y - F_z q$$

- **Green's identities:**

$$\begin{aligned} \int_\Omega (g\Delta f - f\Delta g)\, dx &= \int_{\partial\Omega} (g\partial_n f - f\partial_n g)\, dS \\ \int_\Omega (g\Delta f + \nabla g \nabla f)\, dx &= \int_{\partial\Omega} g\partial_n f\, dS \\ \int_\Omega \Delta f\, dx &= \int_{\partial\Omega} \partial_n f\, dS \end{aligned}$$

where ∂_n is the (outward) normal derivative.

- The **fundamental solution of the Laplace operator** Δ in $\mathbb{R}^n$ is given by the potential

$$K(x) = \begin{cases} (2\pi)^{-1} \log \|x\| & \text{if } n = 2 \\ -(4\pi\|x\|)^{-1} & \text{if } n = 3 \end{cases}$$

- The **Poisson integral formula** is $u(\xi) = \int_{\partial\Omega} H(x, \xi)u(x)dS_x$, where $H(x, \xi)$ is the **Poisson kernel**. The Poisson kernel in the upper half-space in $\mathbb{R}^n$ (that is, $\xi_n > 0$) is

$$H(x', \xi) = \frac{2\xi_n}{\omega_n |x' - \xi|^n} \qquad x' = (x_1, \ldots, x_{n-1})$$

The Poisson kernel for the unit ball in $\mathbb{R}^n$ is

$$H(x, \xi) = \frac{1 - |\xi|^2}{\omega_n |x - \xi|^n} \qquad \|x\| = 1$$

- **Kirchoff's formula** gives the solution to the pure initial value problem for the three dimensional **wave equation** $u_{tt} = c^2\Delta u$ with initial data $u(x, 0) = g(x)$, $u_t(x, 0) = h(x)$.

$$u(x, t) = (4\pi)^{-1} \frac{\partial}{\partial t} \left(t \int_{\|\xi\|=1} g(x + ct\xi)\, dS_\xi \right) + (4\pi)^{-1} t \int_{\|\xi\|=1} h(x + ct\xi)\, dS_\xi$$

- The solution to the pure initial value problem for the **heat equation** $u_t = \Delta u$ with initial condition $u(x, 0) = g(x)$ is given by the convolution $u(x, t) = \int_{\mathbb{R}^n} K(x - y, t)g(y)\, dy$ of the heat kernel $K(x, t)$ with the initial data. The heat kernel for $n = 1$ is given by

$$K(x, t) = (4\pi t)^{-1/2} \exp(-x^2/4t)$$

- The **Fourier transform** $\mathcal{F}g$ and the inverse Fourier transform $\mathcal{F}^{-1}h$ are

$$\mathcal{F}g(\xi) = \int_{\mathbb{R}^n} \exp(-ix \cdot \xi)g(x)\, dx, \qquad \mathcal{F}^{-1}h(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \exp(-ix \cdot \xi)h(\xi)\, d\xi$$

Fourier inversion formula: $\mathcal{F}^{-1}(\mathcal{F}g) = g$. Basic formula: $\mathcal{F}(\partial_k g)(\xi) = i\xi_k \mathcal{F}g(\xi)$.

Do any six problems. Clearly indicate in the table below which problems you want to be graded. If you do not select any problems we will grade the first 6 problems. Good luck!

Problems	1	2	3	4	5	6	7	8

1. Use the method of characteristics to solve the Cauchy problem $u = u_x^2 - 3u_y^2$ with $u(x,0) = x^2$. Is the solution uniquely defined? If so, justify. If not, produce two solutions.

2. Assume that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is sub-harmonic,

$$\Delta u(\mathbf{x}) \geq 0 \quad \text{for all} \quad \mathbf{x} = (x_1, \ldots, x_n) \in \Omega$$

with $\Omega \subset \mathbb{R}^n$ a bounded, connected domain. Show that any such $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies the weak maximum principle

$$\max_{\mathbf{x} \in \overline{\Omega}} u(\mathbf{x}) = \max_{\mathbf{z} \in \partial\Omega} u(\mathbf{z}).$$

3. Consider the initial value problem for a conservation law

$$\begin{aligned} &u_t(x,t) + q'\big(u(x,t)\big)u_x(x,t) = 0 \\ &u(x,0) = g(x) \end{aligned} \tag{1}$$

(a) Use the Leibniz rule

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{a(t)}^{b(t)} u(x,t)\,\mathrm{d}x\right) = u(b(t),t)b'(t) - u(a(t),t)a'(t) + \int_{a(t)}^{b(t)} u_t(x,t)\,\mathrm{d}x$$

to derive the Rankine-Hugoniot jump condition for the speed $s'(t)$ of a shock from the following conservation law property — the solution $u(x,t)$ of (1) must obey

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b u(x,t)\,\mathrm{d}x = q\big(u(a,t)\big) - q\big(u(b,t)\big)$$

for any interval $(a,b) \subset \mathbb{R}$.

(b) Consider following equation

$$u_t + \frac{1}{2}uu_x = 0 \quad x \in \mathbb{R}, t > 0 \qquad u(x,0) = \begin{cases} 2 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x > 0. \end{cases} \tag{2}$$

Find the entropy solution to (2), and justify that your solution is the entropy solution.

4. Consider the hyperbolic equation

$$
\begin{aligned}
u_{tt} - 2\lambda u_{tx} - u_{xx} = 0 \qquad & x \in \mathbb{R}, t > 0 \\
u(x,0) = g(x) \qquad & x \in \mathbb{R}, t = 0, \\
u_t(x,0) = h(x) \qquad & x \in \mathbb{R}, t = 0,
\end{aligned} \tag{3}
$$

for $\lambda \in \mathbb{R}$ any real number. Use an ansatz of the form

$$
u(x,t) = F(x + \lambda_+ t) + G(x + \lambda_- t) \qquad \lambda_\pm := \lambda \pm \sqrt{1 + \lambda^2}
$$

to derive the d'Alembert formula

$$
u(x,t) = \frac{\lambda_+ g(x + \lambda_- t) - \lambda_- g(x + \lambda_+ t)}{\lambda_+ - \lambda_-} + \frac{1}{\lambda_+ - \lambda_-} \int_{x+\lambda_- t}^{x+\lambda_+ t} h(z)\, \mathrm{d}z
$$

for the solution of (3).

5. Solve the following problem —

$$
\begin{aligned}
u_{tt} - u_{xx} &= 0, \quad t > \max\{-x, x\},\ t \geq 0, \\
u(x,t) &= \phi(t), \quad x = t,\ t \geq 0 \\
u(x,t) &= \psi(t), \quad x = -t,\ t \geq 0,
\end{aligned}
$$

where $\phi, \psi \in C^2([0,\infty))$ and $\phi(0) = \psi(0)$.

6. Use the odd extension to find the solution to the following problem

$$
\begin{aligned}
u_t - k u_{xx} &= 0, \quad 0 < x < \infty,\ t > 0, \\
u(x,0) &= f(x), \quad 0 < x < \infty, \\
u(0,t) &= 0, \quad t > 0,
\end{aligned}
$$

where $f \in C([0,\infty))$.

7. Let $\Omega \subset \mathbb{R}^n$ denote a smooth, bounded domain. Suppose that a smooth function $u(\mathbf{x}, t)$ satisfies the heat equation

$$u_t(\mathbf{x}, t) = \Delta u(\mathbf{x}, t)$$

in $\Omega \times \{t > 0\}$, and that either $u(\mathbf{x}, t) = 0$ or $(\partial_{\boldsymbol{\nu}} u)(\mathbf{x}, t) = 0$ on $\partial\Omega$. Use the energy method to prove that

$$E(t) := \frac{1}{2} \int_\Omega u^2(\mathbf{x}, t) \, \mathrm{d}\mathbf{x} + \int_0^t \int_\Omega |\nabla u|^2(\mathbf{x}, s) \, \mathrm{d}\mathbf{x}\mathrm{d}s$$

is constant in time, then prove uniqueness for smooth solutions to non-homogeneous Dirichlet

$$u(\mathbf{x}, 0) = g(\mathbf{x}) \quad \text{and} \quad u(\mathbf{x}, t) = h(\mathbf{x}, t) \quad \text{on } \partial\Omega$$

and non-homogeneous Neumann

$$u(\mathbf{x}, 0) = g(\mathbf{x}) \quad \text{and} \quad (\partial_{\boldsymbol{\nu}} u)(\mathbf{x}, t) = h(\mathbf{x}, t) \quad \text{on } \partial\Omega$$

initial/boundary value problems for the heat equation.

8. Let $\Omega \subset \mathbb{R}^n$ denote a bounded, connected domain with smooth boundary. Let $u(\mathbf{x})$ denote the solution to Poission's equation

$$\Delta u(\mathbf{x}) = f(\mathbf{x}) \quad (\mathbf{x} \in \Omega) \qquad \text{and} \qquad u(\mathbf{x}) = g(\mathbf{x}) \quad (\mathbf{x} \in \partial\Omega),$$

and let $G(\mathbf{x}, \mathbf{y})$ denote the Green's function for Ω. Prove Green's representation

$$u(\mathbf{x}) = \int_{\partial\Omega} (\partial_{\boldsymbol{\nu}} G)(\mathbf{x}, \mathbf{y}) g(\mathbf{y}) \, \mathrm{d}\sigma_{\mathbf{y}} + \int_\Omega G(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) \, \mathrm{d}\mathbf{y}$$

for the solution to Poisson's equation. (Here $(\partial_{\boldsymbol{\nu}} G)(\mathbf{x}, \mathbf{y}) := \nabla_{\mathbf{y}} G(\mathbf{x}, \mathbf{y}) \cdot \boldsymbol{\nu}(\mathbf{y})$ means normal derivative).