Fall 2017

Partial Differential Equation Comprehensive Exam

Do any six problems. Clearly indicate in the table below which problems you want to be graded. If you do not select any problems we will grade the first 6 problems. Good luck!

Problems	1	2	3	4	5	6	7	8

1. Let $\Omega \subset \mathbb{R}^2$ denote the positive quadrant, i.e.

$$\Omega := \left\{ \mathbf{x} = (x, y) \in \mathbb{R}^2 : x > 0,\ y > 0 \right\}.$$

Use the method of reflection to find the Green's function $G(\mathbf{x}, \mathbf{y})$ for Ω, and use it to state a solution to the Dirichlet problem

$$\Delta u(\mathbf{x}) = f(\mathbf{x}) \ \ \mathbf{x} \in \Omega \qquad \text{and} \qquad u(\mathbf{x}) = g(\mathbf{x}) \ \ \mathbf{x} \in \partial\Omega$$

for Poisson's equation.

2. Let $\Omega \subset \mathbb{R}^n$ denote a bounded, open domain, and assume that $u \in C^2(\Omega)$ is twice differentiable. Show that $u(\mathbf{x})$ is harmonic in Ω if and only if $u(\mathbf{x})$ satisfies the mean value property. You may use the identities

$$\int_{B_r(\mathbf{x})} u(\mathbf{z})\,\mathrm{d}\mathbf{z} = \int_0^r \left(\int_{\partial B_s(\mathbf{x})} u(\mathbf{z})\,\mathrm{d}\sigma_{\mathbf{z}} \right) \mathrm{d}s, \quad |\partial B_r(\mathbf{x})| = r^{n-1} |\partial B_1(\mathbf{0})|, \quad |B_r(\mathbf{x})| = \frac{r^n}{n} |\partial B_1(\mathbf{0})|$$

$$\int_{\partial B_r(\mathbf{x})} u(\mathbf{z})\,\mathrm{d}\sigma_{\mathbf{z}} = r^{n-1} \int_{\partial B_1(\mathbf{0})} u(\mathbf{x} + r\mathbf{y})\,\mathrm{d}\sigma_{\mathbf{y}}$$

without proof in your argument.

3. Let $u(x, t)$ denote the solution to the initial value problem

$$\begin{aligned} u_{tt} &= u_{xx} \\ u(x, 0) &= f(x) \\ u_t(x, 0) &= g(x), \end{aligned}$$

where the initial conditions $f \in C^2(\mathbb{R})$ and $g \in C^1(\mathbb{R})$ have compact support. That is, there exists an $R > 0$ so that $f(x) = 0$ and $g(x) = 0$ whenever $|x| \geq R$. Define $k(t)$ and $p(t)$ as

$$k(t) := \int_{\mathbb{R}} u_t^2(x, t)\,\mathrm{d}x \qquad p(t) := \int_{\mathbb{R}} u_x^2(x, t)\,\mathrm{d}x.$$

(a) Show the total energy $E(t) = k(t) + p(t)$ is constant in time.

(b) Show that $k(t) = p(t)$ whenever $t \geq 2R$.

4. Let $\Omega \subset \mathbb{R}^n$ denotes a bounded, connected domain with smooth boundary. Use Green's identity and the energy method to show that $u(\mathbf{x}, t) = 0$ is the unique solution to the following parabolic PDE with bi-harmonic diffusion:

$$\begin{aligned} u_t &= -\Delta(\Delta u) && \mathbf{x} \in \Omega,\ t > 0, \\ \Delta u(\mathbf{x}, t) &= 0 && \mathbf{x} \in \partial\Omega,\ t > 0, \\ u(\mathbf{x}, t) &= 0 && \mathbf{x} \in \partial\Omega,\ t > 0, \\ u(\mathbf{x}, 0) &= 0 && \mathbf{x} \in \Omega,\ t = 0. \end{aligned}$$

5. Suppose that $u \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega}_T)$ solves the heat equation

$$\begin{aligned} u_t(\mathbf{x}, t) &= \Delta u(\mathbf{x}, t) && \text{in} \quad \Omega_T := \{(\mathbf{x}, t) : \mathbf{x} \in \Omega,\ 0 < t < T\} \\ u(\mathbf{x}, t) &= h(\mathbf{x}, t) && \mathbf{x} \in \partial\Omega,\ t > 0 \\ u(\mathbf{x}, 0) &= g(\mathbf{x}) && \mathbf{x} \in \Omega,\ t = 0 \end{aligned} \tag{1}$$

in $\Omega \subset \mathbb{R}^n$ a bounded domain with smooth boundary. Let

$$\partial_{\mathrm{p}}\Omega_T := \{(\mathbf{x}, t) \in \overline{\Omega}_T : \mathbf{x} \in \partial\Omega \ \text{ or } \ t = 0\}$$

denote the parabolic boundary.

(a) State and prove the weak maximum principle for the heat equation.

(b) Let $\Omega = B_1(\mathbf{0})$ denote the unit ball in $\mathbb{R}^2$ centered at the origin. Show the solution to (1) satisfies the inequality

$$\mathrm{e}^{-8t}\left(1 - |\mathbf{x}|^2\right)^2 \leq u(\mathbf{x}, t) \leq \mathrm{e}^{-4t}(1 - |\mathbf{x}|^2)$$

if $g(\mathbf{x}) = 1 - |\mathbf{x}|^2$ and $h(\mathbf{x}, t) = 0$. You may use the identities $\Delta|\mathbf{x}|^2 = 4$ and $\Delta|\mathbf{x}|^4 = 16|\mathbf{x}|^2$ (valid in two dimensions) without proof.

6. Consider the pure initial value problem for the damped wave equation,

$$u_{tt} + \alpha u_t = u_{xx} \qquad u(x, 0) = g(x) \ \ u_t(x, 0) = h(x), \tag{2}$$

for $\alpha > 0$ a drag coefficient.

(a) Fix a point $x_0 \in \mathbb{R}$ and a time $t_0 > 0$. For $0 \leq t \leq t_0$ let

$$E(t) := \frac{1}{2} \int_{x_0 - (t_0 - t)}^{x_0 + (t_0 - t)} u_t^2(x, t) + u_x^2(x, t)\, \mathrm{d}x$$

denote the total energy (kinetic plus potential) in the interval $(x_0 - (t_0 - t), x_0 + t_0 - t)$. Use Leibniz rule and the PDE (2) to show that

$$E'(t) \leq 0$$

for $0 < t < t_0$, and so $E(t)$ is non-increasing.

(b) Suppose $g(x) = h(x) = 0$ in the interval $(x_0 - t_0, x_0 + t_0)$. Show that

$$u(x,t) = 0$$

in the entire triangular region $T := \{(x,t) : x_0 - (t_0 - t) \leq x \leq x_0 + (t_0 - t),\ 0 \leq t \leq t_0\}$.

(c) Show that solutions of the initial value problem (2) are unique.

7. Use the method of characteristics to find a solution to the quasi-linear PDE

$$u_x u_y = u \qquad \text{with} \quad u(x,y) = y^2 \quad \text{on} \quad \Gamma := \{(x,y) : x = 0\}.$$

Is the solution uniquely defined near $(x,y) = (0,0)$? If so, state the solution. If not, find another solution.

8. Find the entropy solution to Burger's equation

$$u_t + uu_x = 0 \quad x \in \mathbb{R},\ t > 0$$

$$u(x,0) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad 0 < x < 1 \\ 0 & \text{if} \quad x > 1. \end{cases}$$

Sketch the characteristics in the (x,t) plane, and clearly indicate any shocks that occur.