Applied Nonlinear ODE Masters Exam – Sample 3

Do any 6 problems. Clearly indicate which one you want to be graded. Good luck!

1. Suppose that, for the two plane systems $d\vec{x}/dt = \vec{X}(\vec{x})$ and $d\vec{y}/dt = \vec{Y}(\vec{y})$, and for a given closed curve Γ, there is no point on Γ at which $\vec{X}$ and $\vec{Y}$ are opposite in direction. Show that the index of Γ is the same for both systems.

   The system $x' = y^2$, $y' = xy$ has a saddle point at the origin. Show that the index of the origin for the system $x' = y^2 + \varepsilon x^2$, $y' = xy - \varepsilon y^2$ is likewise -1 for $|\varepsilon|$ small enough.

2. Use the harmonic balance method to find the amplitude and frequency of the limit cycle of the equation $x'' + \varepsilon(|x| - 1)|x'|x' + x + \varepsilon x^3 = 0$.

3. Use the Lindstedt method to find a perturbation solution to order ε^2 in the expansion of the periodic solution of $x'' + x + \varepsilon x^2 = 0$ with the initial condition $x(0) = a_0$ and $x'(0) = 0$.

4. Investigate the equilibrium points of

$$x'' + \varepsilon(\alpha x^4 - \beta)x' - x + x^3 = 0, \quad (\alpha, \beta > 0)$$

   for $0 < \varepsilon \ll 1$. Use the perturbation method to find the approximate value of β/α at chich homoclinic paths exist.

5. Prove that for the regular linear system $d\vec{x}/dt = A(t)\vec{x}$, every solution is Liapunov stable on $t \geq t_0$, t_0 arbitrary, if and only if every solution is bounded as $t \to \infty$.

6. Suppose that in a neighborhood $\mathcal{N}$ of the origin, (i) $\vec{x}' = \vec{X}(\vec{x})$ is a regular system and $\vec{X}(\vec{0}) = \vec{0}$; (ii) $V(\vec{x})$ is continuous and positive definite; and (iii) $dV(\vec{x})/dt$ is continuous and negative semidefinite. Show that the zero solution of the system is uniformly stable.

7. For the system

$$x' = y, \quad y' = f(x, y), \quad (f(0, 0) = 0),$$

   show that V given by

$$V(x, y) = \frac{1}{2}y^2 - \int_0^x f(u, 0)du$$

   is a weak liapunov function for the zero solution when, in some neighborhood of the origin,

$$[f(x, y) - f(x, 0)]y \leq 0, \quad \int_0^x f(u, 0)du < 0.$$

8. Investigate the bifurcations of the system

$$x' = 2x(\mu - x) - (x + 1)y^2, \quad y' = y(x - y)$$

   where μ is a parameter.