Applied Nonlinear ODE Masters Exam – Sample 2

Do any 6 problems. Clearly indicate which one you want to be graded. Good luck!

1. Consider $x' = y$ and $y' = f(x, y, \lambda)$, where f and f' are conitnuous.
   (a) Show that the index I_Γ of any simple closed curve Γ that encloses all equilibrium points can only be -1, 1 or zero.
   (b) Show that at a bifurcation point the sum of the indices of the equilibrium points resulting from the bifurcation is unchanged.
   (c) Deduce that the system $x' = y$, $y' = -\lambda x + x^3$ has a saddle point at $(0, 0)$ when $\lambda < 0$, which bifurcates into a stable spiral or node and two saddle points as λ becomes positive.

2. (a) For the system $x' = X(x, y)$, $y' = Y(x, y)$, show that there are no closed paths in a simply connected region in which $\partial(\rho X)/\partial x + \partial(\rho Y)/\partial y$ is of one sign, where $\rho(x, y)$ is any function having continuous partial derivatives.
   (b) By using the result in (a) with $\rho = e^{-2x}$ show that the following system has no periodic solution
   $$x' = y, \quad y' = -x - y + x^2 + y^2.$$

3. Consider $x' = y(2y^2 - 1)$, $y' = 2x(x^2 - 1)$.
   (a) Classify all the equilibrium points of the system according to their linear approximations.
   (b) Find the homoclinic paths.
   (c) Show that the heteroclinic paths lie on the circle $x^2 + y^2 = 3/2$ and the hyperbola $x^2 - y^2 = 1/2$.
   (d) Sketch the phase diagram.

4. Consider $x'' + \varepsilon h(x, x') + g(x) = 0$. Let $g(0) = 0$ and g be continuous and strictly increasing in $|x| < \delta$.
   (a) Show that the origin for the equation $x'' + g(x) = 0$ is a center.
   (b) Find the potential energy V and the total energy $\mathcal{E}(t)$.
   (c) Let $\zeta(t, a)$ be its periodic solutions near the origin, where a is a constant parameter, say amplitude, and let $T(a)$ be the period. By using the total energy, show that the periodic solutions of the original equation satisfy
   $$\int_0^{T(a)} h(\zeta, \zeta')\zeta' dt = 0.$$

5. Consider $x'' + x^3 = 0$.
   (a) Substitute $x = a \cos \omega t$ to find the frequency-amplitude relation.
   (b) Construct the associated linear equation by finding the Fourier series of x^3 (ignore the higher harmonic terms).

(c) Show how the process (a) and (b) are equivalent.

6. Apply Lindstedt's method to van der Pol's equation $x''+\varepsilon(x^2-1)x'+x=0$. Find the frequency of the limit cycle to order ε^2. Derive a perturbation solution to order ε.

7. If A is a constant $n \times n$ matrix and the eigenvalues of A have negative real parts, $\vec{C}(t)$ is continuous for $t \geq t_0$ and $\int_{t_0}^{t} \|\vec{C}(s)\| ds$ is bounded for $t > t_0$, where $\|\vec{x}\| = \sqrt{\sum_{i=1}^{n} |x_i|^2}$. Then show that all solutions of $\vec{x}' = (A+\vec{C}(t))\vec{x}$ are asymptotically stable.

8. Let
$$x' = -ax + bf(y), \quad y' = cx - df(y),$$
where $f(0)=0$, $yf(y)>0$ for $y \neq 0$, and $bc - ad < 0$, where a, b, c, d are positive. Show that the system is asymptotically stable, that is, show that for suitable values of p and q,
$$V = \frac{px^2}{2} + q\int_0^y f(u)du$$
is a strong Liapunov function for the zero solutions.