Applied Nonlinear ODE Masters Exam – Sample 1

Do any 6 problems. Clearly indicate which one you want to be graded. Good luck!

1. Consider $x' = f(x,y)$, $y' = g(x,y)$. Suppose that Γ lies in a simply connected region on which f, g and their first derivatives are continuous and f, g are not simultaneously zero (i.e. no equilibrium point there). Show that the index I_Γ is zero.

2. (a) For the system $x' = X(x,y)$, $y' = Y(x,y)$, $curl(X,Y) = \vec{0}$ in a simply connected region $\mathcal{D}$. Show that the system has no closed paths in $\mathcal{D}$.

   (b) By using the result in (a) show that the following system has no periodic solution

$$x' = y + 2xy, \quad y' = x + x^2 - y^2.$$

3. Consider $x' = a - x^2$, $y' = -y + (x^2 - a)(1 - 2x)$.

   (a) Classify all the equilibrium points of the system according to their linear approximations.

   (b) Verify that the phase diagram has a saddle-node connection given by $y = x^2 - a$.

   (c) Sketch the phase diagram.

4. Consider $x'' + \varepsilon h(x^2 + x'^2 - 1)x' + x = 0$ where h is continuous, $h(u) < 0$ for $u < 0$, $h(0) = 0$, and $h(u) > 0$ for $u > 0$.

   (a) Show that the problem has the periodic solution $x = \cos(t + \alpha)$ for any α.

   (b) Show that this solution is a stable limit cycle when $\varepsilon > 0$.

5. Apply the method of harmonic balance to the equation

$$x'' + x - \alpha x^2 = 0, \quad \alpha > 0,$$

   using the approximate form of solution $x = c + a\cos(\omega t)$ to show that, for α small,

$$\omega^2 = 1 - 2\alpha c, \quad c = \frac{1}{2\alpha}(1 - \sqrt{1 - 2\alpha^2 a^2}).$$

   Deduce the frequency-amplitude relation

$$\omega = (1 - 2\alpha^2 a^2)^{1/4}, \quad a < 1/\sqrt{2}\alpha.$$

6. Consider $x'' + \Omega^2 x + \varepsilon f(x) = \Gamma \cos t$, where Ω is not close to an odd integer, and $f(x)$ is an odd function of x with expansion

$$f(a\cos t) = -A_1(a)\cos t - A_3(a)\cos 3t - \dots$$

   Derive a perturbation solution of period 2π, to oder ε.

7. Prove that Liapunov stability implies Poincaré statbility of the solution. Show also that the converse is not true by showing a counterexample.

8. For the system

$$x' = f(x) + by, \quad y' = cx + dy, \quad (f(0) = 0),$$

show that V given by

$$V(x, y) = (dx - by)^2 + 2d \int_0^x f(u)du - bcx^2$$

is a strong Liapunov function for the zero solution when, in some neighborhood of the origin,

$$d\frac{f(x)}{x} - bc > 0, \quad \frac{f(x)}{x} + d < 0, \quad \text{for} \quad x \neq 0.$$

Deduce that for initial conditions in the circle $x^2 + y^2 < 1$, the solutions of the system

$$x' = -x^3 + x^4 + y, \quad y' = -x$$

tend to zero.