Master's Comprehensive Exam – Statistical Inference NAME: ___________________________

Instructions: Solve *seven* out of the *ten* problems given below.

Circle *seven* problems chosen: 1 2 3 4 5 6 7 8 9 10

September 13, 2014

1. Let $X_1, X_2, \dots, X_n$ be a random sample of size n from a $N(0, \sigma^2)$ distribution.
    a. Show that $S^2 = \sum_{i=1}^{n} X_i^2 \,/n$ is a complete sufficient statistic.
    b. Show that $[X_1/S, X_2/s, \dots, X_n/S]$ is independent of S^2.

2. Let $X_1, X_2, \dots, X_n$ be a random sample of size n from an Exponential distribution with pdf
$$f(x|\theta) = \theta \exp(-\theta x)\, I_{(0,\infty)}(x).$$
    a. Find the distribution of $T = \sum_{i=1}^{n} X_i$.
    b. Find the UMVUE of θ.

3. Consider a sequence of random variables defined on the same probability space where
$$P\left(X_n = \frac{1}{n}\right) = P\left(X_n = -\frac{1}{n}\right) = \frac{1}{2} \text{ and } P(X = 0) = 1$$
    a. Show that X_n converges in distribution to a random variable X. Also find the cdf of X.
    b. Prove or disprove that $X_n \xrightarrow{p} X$.

4. Let $X_{1:4} < X_{2:4} < \cdots < X_{4:4}$ denote the order statistics of a random sample of size 4 from the $\text{Uniform}(0, \theta)$. Let the observed value of $X_{4:4}$ be X_4. We reject $H_0 \colon \theta = 1$ and accept $H_1 \colon \theta \neq 1$ if either $x_4 > 1$ or $x_4 \leq 1/2$. Find the power function $\gamma(\theta)$ of the test, $\theta > 0$.

5. Verify that the distribution of the likelihood ratio statistic under $N(\mu, \sigma^2)$ sampling is exactly χ_1^2.

6. Let $X_1, X_2, \dots, X_n$ be a random sample from $N(0, \sigma^2)$.
    a. Find the UMVUE for σ^2.
    b. Show that your answer to part a) is statistically independent to $X_{(1)}/X_{(2)}$, the ratio of the first two order statistics. Hint: You shouldn't have to work too hard here

7. Let $X_1, X_2, \dots, X_n$ be a random sample from $\text{Uniform}[0, \theta]$, where $\theta > 0$. Let $Z_n = n(\theta - X_{nn})$, where $X_{nn} = \max\{X_i\}$. Find the limiting distribution of Z_n.

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8. Prove the followings:
   Let (X, Y) be random variables, then
   a. $E(X) = E(E(X|Y))$
   b. $Var(X) = Var\big(E(X|Y)\big) + E(Var(X|Y))$

   Now, using the above to solve the following question:
   A quality control plan for an assembly line involves sampling $n = 100$ finished items per day and counting Y, the number of defectives. If p denotes the probability of observing a defective, which varies from day to day and is assumed to have a Beta distribution with $(\alpha = 2, \beta = 1)$.
   c. What is the expected number of defectives for any given day?
   d. What is the standard deviation of number of defectives for any given day?

9. Solve the following two parts: (Hint: Use Complete Sufficient Statistic)
   a. Suppose $X_1, X_2, \ldots, X_n$ are iid Poisson distribution with mean λ. Find the UMVUE of λ^2 and the UMVUE of $(\lambda - 1)(\lambda - 1)$.
   b. Suppose $X_1, X_2, \ldots, X_n$ are iid $\text{Uniform}[0, \theta]$. Assume $n \geq 3$. Find UMVUE of θ^3 and UMVUE of $1/\theta^2$.

10. Solve the following two parts:
    a. Let $X_1, X_2, \ldots, X_n$ be iid with one of two pdfs. If $\theta = 0$,
    $$f(x|\theta) = \begin{cases} 1, & \text{for } 0 < x < 1 \\ 0, & \text{Otherwise} \end{cases}$$
    If $\theta = 1$,
    $$f(x|\theta) = \begin{cases} \dfrac{1}{2\sqrt{x}}, & \text{for } 0 < x < 1 \\ 0, & \text{Otherwise} \end{cases}$$
    Find MLE of θ.
    b. Let the random element X be distributed as P_θ where $\theta \in \{\theta_0, \theta_1\}$. Let α and β denote the Type I and Type II error probabilities of a test for testing the simple null hypothesis θ_0 versus the simple alternative θ_1. Instead of the Neyman-Pearson approach of maximizing $1 - \beta$ subject to a fixed α, another reasonable approach is to minimize a linear combination of α and β. Obtain the structure of a test $\phi^*(x)$, in terms of the likelihood ratio, that minimizes $2\alpha + \beta$. (Hint: The Neyman-Pearson lemma does not apply. Solve directly.)

STOP HERE!!