California State University, Long Beach
Department of Mathematics
Complex Analysis Comprehensive Examination
February 4, 2023

Do three problems from each part, for a total of six problems. Circle the problems you want to be graded.

PART A

Problem 1. (a). Compute

$$\int_C z^2 dz,$$

where C is a smooth curve from $\sqrt[3]{3}e^{i\pi/6}$ to $\sqrt[3]{6}e^{i\pi/3}$.

(b). Compute

$$\int_C \frac{z}{z^2-1} dz,$$

where C is the triangle with vertices -2, $-i$ and i, going counterclockwise.

Problem 2. (a). Find the Laurent series around $z=0$ for $f(z)=\frac{z}{z-1}$ on $\{z : 0<|z|<1\}$.

(b). Find the Laurent series around $z=0$ for $f(z)=\frac{z}{z-1}$ on $\{z : 1<|z|<\infty\}$.

Problem 3. Evaluate the integral

$$\int_0^\infty \frac{z^2}{z^4+z^2+1}\, dz.$$

Problem 4. Recall a Möbius transformation is a function of the form

$$f(z)=\frac{az+b}{cz+d}$$

on the extended complex plane $\mathbb{C}\cup\{\infty\}$.

(a). Show that every Möbius transformations has either one or two fixed points on $\mathbb{C}\cup\{\infty\}$ (p is a fixed point of f if $f(p)=p$).

(b). Find all those Möbius transformations f such that 1 is its only fixed point.

Problem 5. (a). State Rouché's theorem.

(b). Prove that $2z^5-6z-1$ has one root in $\{z : |z|<1\}$ and four roots in $\{z : 1\le|z|\le 2\}$.

PART B

Problem 6. Let f be analytic on a domain D, and suppose $f(D)$ is a subset of a line. Prove that f is constant.

Problem 7. Let f be analytic on the unit disc $D = \{z : |z| \leq 1$. Assume $|f(z)| = |e^z|$ for all z with $|z| = 1$. Find all such f.

Hint: the maximum modulus principle.

Problem 8. Assume f is entire (i.e. analytic on $\mathbb{C}$), and there is a nonempty disc such that f does not attain any values in the disc. Prove that f is constant.

Problem 9. Assume f is entire (i.e. analytic on $\mathbb{C}$), satisfying

$$\lim_{z \to \infty} \frac{f(z)}{z} = 0.$$

Prove that f is a constant.

Problem 10. Consider the polynomial

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0.$$

Let $D = \{z : |z| \leq 1\}$ be the closed unit disc. Prove that

$$\sup_{z \in D} |f(z)| \geq |a_n|.$$

Hint: Rouché's theorem.