California State University, Long Beach
Department of Mathematics and Statistics

Complex Analysis Comprehensive Examination

February 12, 2022

On the first page of your work, please write the numbers of the problems that you want graded.

Do three problems from each part, for a total of six problems.

PART A

1. (a) Let $u : D \mapsto \mathbb{R}$, where D denotes the unit disk. Define what it means for u to be harmonic, and define what it means for u to have a harmonic conjugate, $v : D \mapsto \mathbb{R}$.

   (b) Let $u(x, y) = xe^x \cos y - ye^x \sin y + x^3 - 3xy^2 + y + 1$ for $(x, y) \in D$. If they exist, find all harmonic conjugates of u.

2. (a) Classify the singularity of the function $f(z) = e^{1/z}$ at $z = 0$.

   (b) Evaluate $\int_\Gamma e^{1/z} dz$, where Γ is the unit circle $|z| = 1$ traversed once counterclockwise.

3. Let n be a positive integer and $\omega = e^{i2\pi/n} = \cos(2\pi/n) + i \sin(2\pi/n)$.

   (a) Show that $1 + \omega^k + \omega^{2k} + \cdots + \omega^{(n-1)k} = 0$ for any integer k which is not a multiple of n.

   (b) Define an $n \times n$ matrix $A = (a_{pq})$ by $a_{pq} = \omega^{pq} = e^{i2\pi pq/n}$, $1 \leq p, q \leq n$. Find A^{-1}.

4. Consider the inversion $f(z) = 1/z$ and denote its real and imaginary parts by u and v.

   (a) Find the families of level curves of $u(x, y) = \alpha$ and $v(x, y) = \beta$ where $\alpha, \beta \in \mathbb{R}$. Prove that at a point of intersection two level curves meet orthogonally.

   (b) Determine the image under f of the circle $|z - z_0| = r$ where $z_0 \in \mathbb{C}$ and $r > 0$.

5. Compute the integral $\displaystyle\int_0^\infty \frac{dx}{x^4 + 1}$.

PART B

6. Suppose $f(z)$ is analytic on $\{z : 0 < |z| < 1\}$ and $|f(z)| \leq e^{|z|} \log^2(1/|z|)$.

   (a) Show that f has a removable singularity at 0.

   (b) Show that f is identically 0.

7. Let $f(z) = z^6 + 6z - 1$. Prove that f has two real roots on the interval $-2 < x < 2$ and four (non-real) complex roots in the annulus $1 < |z| < 2$.

8. Let h be analytic on the closed unit disk $\{z : |z| \leq 1\}$, and suppose the image of the unit circle $\{z : |z| = 1\}$ is contained in the open unit disk. Prove that h has exactly one fixed point in the open unit disk $\{z : |z| < 1\}$. A fixed point z_0 of h is by definition a point which satisfies $z_0 = h(z_0)$.

9. (a) Let $f(z) = u(x, y) + iv(x, y)$ be an entire function satisfying $u(x, y) \leq x$ for all $z = x + iy$. Show that $f(z)$ is a polynomial of degree at most one.

   (b) Prove that if f is entire and $|f(z)| \to \infty$ as $|z| \to \infty$, then f must have at least one zero.

10. Let $p(z) = \sum_{k=0}^{n} a_k z^k$, $a_n \neq 0$, be a polynomial of degree n such that $|p(z)| \leq M$ for $|z| \leq R$. Show that $|p(z)| \leq M(|z|/R)^n$ for $|z| \geq R$.