CALIFORNIA STATE UNIVERSITY, LONG BEACH
DEPARTMENT OF MATHEMATICS AND STATISTICS

FALL 2022
COMPLEX ANALYSIS COMPREHENSIVE EXAMINATION
(September 17, 2022)

Complete three problems from each part for a total of six problems. Circle the problems you want to be graded.

PART A

1. a. Prove that the function $f(z) = \bar{z}$ is not analytic.
   b. Assume f is analytic. Prove $g(z) = \overline{f(\bar{z})}$ is analytic.

2. a. Prove that all roots of $z^8 + 2x^7 + 5z + 1$ satisfy $|z| < 3$.
   b. Let γ be the circle of radius 3 about 0 traversed once in the counterclockwise direction. Compute
   $$\int_\gamma \frac{8z^7 + 14z^6 + 5}{z^8 + 2z^7 + 5z + 1}\,dz.$$
   c. Again, let γ be the circle of radius 3 about 0 traversed once in the counterclockwise direction. Compute
   $$\int_\gamma \frac{8z^8 + 14z^7 + 5z}{z^8 + 2z^7 + 5z + 1}\,dz.$$

3. Let $f(z) = \frac{1}{z(z-1)(z-2)}$. Find the Laurent expansion of f in the annulus $\{z \in \mathbb{C} | a < |z| < 2\}$.

4. Compute $\int_0^\infty \frac{1}{1+x^4}\,dx$.

5. a. Find a conformal map that maps the first quadrant $\{x + iy |\ x > 0 \text{ and } y > 0\}$ to the unit disc $B_1(0) = \{x + iy |\ x^2 + y^2 < 1\}$.
   b. Find a conformal map that maps $\{x + iy |\ 0 < x < 1\}$ to the unit disc.

PART B

6. Let $\langle z_n \rangle$ be a nonzero sequence contained in the unit disk $B_1(0) = \{x + iy |\ x^2 + y^2 < 1\}$, and suppose $\lim_{n\to\infty} z_n = 0$. Prove that if f is analytic on $B_1(0)$, and $f(z_n) = 0$ for all $n \in \mathbb{N}$, then f is constant on $B_1(0)$.

7. Let f and g be entire functions such that, for all $z \in \mathbb{C}$ for which $Re\ f(z) \le kRe\ g(z)$, for some real constant $k > 0$. Prove there exists $a, b \in \mathbb{C}$ such that $f(z) = ag(z) + b$. [Hint: consider $e^{h(z)}$ for some function $h(z)$, and use Liouville's Theorem.]

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8. a. State and prove Morera's Theorem.
   b. Give the statement of a claim or theorem whose proof relies on Morera's Theorem (you need not provide the proof).

9. Let R be the radius of convergence of the power series $\sum_{k=0}^{\infty} a_k z^k$. Prove the Cauchy-Hadamard formula:

$$R = \frac{1}{\lim\limits_{k\to\infty} \sqrt[k]{a_k}}$$

assuming this limit exists.

10. a. Evaluate $\int_\gamma e^{1/z}dz$, where γ is the unit circle traversed once in the counterclockwise direction.
    b. Prove there is no sequence of polynomials that converges uniformly to $e^{1/z}$ on the unit circle.