California State University, Long Beach

Department of Mathematics and Statistics

Complex Analysis Comprehensive Examination

February 20th , 2021

On the first page of your work, please write the numbers of the problems that you want graded.

Do three problems from each part, for a total of six problems.

PART A

A1. Suppose f is analytic in a region Ω and $\operatorname{Re} f = \operatorname{Im} f$. Use Cauchy-Riemann equations to show that f must be a constant in Ω.

A2. (a) Classify the singularities of the function $e^{1/z}$ at $z = 0$.

(b) Evaluate $\int_{\Gamma} e^{1/z}\,dz$, where Γ is the unit circle, traversed once counterclockwise.

A3. Let $f(z) = \dfrac{1}{z^2 + 2z}$. Expand f in its Laurent series that is centered at zero and that converges when $z = -3$.

A4. Show that $z^4 + 2z^2 - z + 1$ has exactly one root in the first quadrant. Use the argument principle.

A5. Evaluate $\int_C \left(z^4 + 1\right)^{-1} dz$. The contour C is the imaginary axis, oriented upwards.

PART B

B1. Let f be entire. Prove that if f has infinitely many distinct zeros in the open unit disk, then $f(z)$ is identically 0.

B2. Use Liouville's theorem to prove the following statement. If $f(z)$ is entire and there is a nonempty disk such that $f(z)$ does not attain any value in the disk, then $f(z)$ is constant.

B3. Show that a Moebius transformation has either one or two fixed points on the extended complex plane $\mathbb{C}^*$. Which ones have ∞ as a fixed point? Prove your answer.

B4. Suppose that $f(z)$ is analytic on $\{|z|<1\}$ and continuous on $\{|z|\leq 1\}$. Suppose that $f(z)$ is real-valued for all $|z|=1$. Show that f is constant for all $|z|\leq 1$. [Hint: one way to do this would be to consider $g(z)=e^{i f(z)}$.]

B5. Let $f(z)$ and $g(z)$ be analytic in a bounded region D and continuous on $D\cup\partial D$. If $f(z)=g(z)$ on the boundary ∂D, show that $f(z)=g(z)$ on D.