California State University, Long Beach
Department of Mathematics
Complex Analysis Comprehensive Examination
February 9, 2019

Do three problems from each part, for a total of six problems. Circle the problems you want to be graded.

PART A

1. Find the Laurent series of the function $f(z) = \frac{1}{(z-1)(z-2)}$ around zero, on the following domains:

i). $\{z : |z| < 1\}$.
ii). $\{z : 1 < |z| < 2\}$.
iii). $\{z : |z| > 2\}$.

2. Find a (one-to-one and onto) conformal map (write its explicitly) that maps the upper half disc $\{z : |z| < 1\} \cap \{z : \text{Im}(z) > 0\}$ to the upper half plane $\{z : \text{Im}(z) > 0\}$. Here $\text{Im}(z)$ is the imaginary part of z, i.e. $\text{Im}(x + iy) = y$.

3. Evaluate the integral

$$\int_\gamma \frac{1}{z^3 + 1}\, dz,$$

where γ is the imaginary axis traversed in the upward direction. Justify your answer.

4. Evaluate

$$\int_{-\infty}^{\infty} e^{-x^2} e^{-itx}\, dx,$$

for $t \in \mathbb{R}$. Use that the value of this integral at $t = 0$ is $\sqrt{\pi}$. (Hint: Integrate $\exp(-z^2)$ around the rectangle with vertices $\pm R$ and $(it/2) \pm R$. Next, let R go to infinity.)

5. Prove that $2z^5 + 6z - 1$ has one root in the interval $0 < x < 1$ and four roots in the annulus $1 \leq |z| \leq 2$.

PART B

6. Let $f(z) = e^z/z$, where z ranges over the annulus $\frac{1}{2} \leq |z| \leq 1$. Find the points where the maximum and minimum values of $|f(z)|$ occur and determine these values.

7. State and prove the argument principle for a meromorphic function $f(z)$ and a curve γ which bounds an open, bounded, and connected set D.

8. Suppose $f(z)$ is analytic on $\{z : 0 < |z| < 1\}$ and $|f(z)| \leq \log(1/|z|)$.

(a) Show that f has a removable singularity at 0.
(b) Show that f is identically 0.

9. (a) Use Cauchy's estimates to prove that if f is entire and bounded, then f is constant.
(b) Assume f is entire and there are $\epsilon, C > 0$, so that for all z we have

$$|f(z)| \leq C(1 + |z|)^{1-\epsilon}.$$

Prove that f is constant.
(c) Prove that if f is entire and $f(z) \to \infty$ as $|z| \to \infty$, then f must have at least one zero.

10. Let $\zeta(z) = \sum_1^\infty n^{-z}$, where $n^{-z} = \exp(-z \log n)$ and $\log n$ is the natural logarithm of n.
(i) Show that the series converges absolutely for $\mathrm{Re}(z) > 1$;
(ii) Show that $\zeta(z)$ is analytic in $\mathrm{Re}(z) > 1$. Here $\mathrm{Re}(z)$ is the real part of z, i.e. $\mathrm{Re}(x + iy) = x$.