California State University, Long Beach
Department of Mathematics

FALL 2018
COMPLEX ANALYSIS COMPREHENSIVE EXAMINATION
(September 15, 2018)

Do three problems from each part, for a total of six problems. Circle the problems you want to be graded.

PART A

1. Let f be analytic on an open set U containing 0, and assume that $f'(0) \neq 0$. Show that
$$\frac{2\pi i}{f'(0)} = \oint_C \frac{1}{f(z) - f(0)}\, dz,$$
where C is a small circle centered at 0.

2. (a) State Rouche's theorem.
   (b) Find the number of zeros of the polynomial
   $$p(z) = 2z^5 + 6z - 1$$
   that lie inside the annulus $\{z : 1 < |z| < 2\}$.
   (c) Prove Rouche's theorem by applying the Argument Principle.

3. Assume that $a > b > 0$. Evaluate using residue theory:
$$\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta}$$

4. Let $f(z)$ be a rational function with no poles on the real axis. Assume that the degree of the denominator is at least two more than the degree of the numerator. Give a formula for $\int_{-\infty}^{\infty} f(x)\, dx$ in terms of the residues of f. Prove your answer by applying the residue theorem.

5. Evaluate
$$\int_{-\infty}^{\infty} \exp(-x^2/2)\exp(-itx)\, dx,$$
for $t \in \mathbf{R}$. Use that the value of this integral at $t = 0$ is $\sqrt{2\pi}$. (Hint: Integrate $\exp(-z^2/2)$ around the rectangle with vertices $\pm R$ and $it \pm R$. Next, let R go to infinity.)

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PART B

1. (i) Let $f(z)$ be an entire function. Show that if there is a disk $\{z : |z-a| < r\}$, $r > 0$, $a \in \mathbf{C}$, such that $f(z)$ does not attain any values in this disk, then $f(z)$ is constant.
   (ii) Prove or disprove: If f is entire and $\{a\} \cap f(\mathbf{C}) = \emptyset$ for some $a \in \mathbf{C}$, then f is constant.

2. Let z_1 and z_2 be two distinct points in $\mathbf{C} \cup \{\infty\}$ that are symmetric with respect to a circle Γ. Prove that every circle containing both z_1 and z_2 is orthogonal to Γ. [*Hint*: Map the circles onto lines.]

3. Prove that if a Mobius transformation f has exactly one fixed point in the extended complex plane $\mathbf{C} \cup \{\infty\}$, then it is conjugate to the translation $g(z) = z + 1$. That is, show that there exists a Mobius transformation h such that $g = h \circ f \circ h^{-1}$.

4. Let U be a bounded open connected set in the complex plane. Let $f_1(z), f_2(z), f_3(z), \ldots$ be a sequence of functions, each of which is continuous in the closure of U and analytic in U. Show that if the sequence $f_1(z), f_2(z), f_3(z), \ldots$ converges uniformly on the boundary of U then it converges uniformly on U.

5. Let $f(z)$ be a bounded analytic function on the right half-plane. Suppose that $f(z)$ extends continuously to the imaginary axis and satisfies $|f(iy)| \leq M$ for all points iy on the imaginary axis. Show that $|f(z)| \leq M$ for all z in the right half-plane. *Hint.* For $\epsilon > 0$ small, consider
$$(z+1)^{-\epsilon} f(z)$$
on a large semidisk.