Spring 2024 – Algebra Comprehensive Exam Name: ____________

Choose six problems total, including at least two from Part I and two from Part II. Enter the numbers of the problems you want graded here:

Problems							Total
Scores							

Part I: Groups (Choose at least two.)

1. Let H and K be finite groups. Let $G = H \times K$.
   (a) If $h \in H$ has order m and $k \in K$ has order n, what is the order of (h, k) in G? Justify your answer.
   (b) How many elements of order 20 are there in the group $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/10\mathbb{Z})$?

2. In each item below you are given a group G with a subgroup H. Determine if H is a normal subgroup of G. Justify your answers.
   (a) G is a finite group with a unique element b of order 2; $H = \langle b \rangle$.
   (b) $G = S_4$; $H = \langle (123) \rangle$.
   (c) $G = D_{12}$, the dihedral group of order 12; H is a Sylow 2-subgroup of G.

3. (a) Let $p < q < r$ be prime numbers. Prove that there are no simple groups of order pqr.
   (b) How many isomorphism classes of abelian groups of order 360 are there? For each one give both its invariant factor decomposition and its elementary divisor decomposition.

4. Let G be a group. Prove or disprove each of the following statements.
   (a) If every finitely generated subgroup of G is abelian, then G is abelian.
   (b) If every subgroup of G is normal, then G is abelian.
   (c) If A and B are normal subgroups of G with G/A and G/B abelian, then $G/(A \cap B)$ is abelian.
   (d) If every element of G has finite order, then G is finite.

5. If $A = \{a_1, a_2, \ldots, a_n\}$ is a set, let S_A denote the permutations of $\{a_1, a_2, \ldots, a_n\}$ and let S_n denote $S_{\{1,2,\ldots,n\}}$.
   (a) Let φ be an automorphism of S_3. Show that φ will map any transposition $\tau \in S_3$ to another transposition.
   (b) Show that if φ fixes each transposition, then φ is the identity automorphism.
   (c) Show that $\text{Aut}(S_3) \simeq S_A$, where A is the set of transpositions in S_3.

Part II: Rings and Linear Algebra (Choose at least two.)

6. Let R and S be commutative rings with identity and let $\phi : R \to S$ be a ring homomorphism such that $\phi(1_R) = 1_S$. For each part below, prove it or give a counterexample.
   (a) If I is an ideal of S, then $\phi^{-1}(I) = \{ r \in R \mid \phi(r) \in I \}$ is an ideal of R.
   (b) If J is an ideal of R, then $\phi(J)$ is an ideal of S.
   (c) If P is a prime ideal of S, then $\phi^{-1}(P)$ is a prime ideal of R.
   (d) If M is a maximal ideal of S, then $\phi^{-1}(M)$ is a maximal ideal of R.

7. Let R be a <u>noncommutative</u> ring with identity. Define
$$Z(R) = \{ r \in R \mid rs = sr \forall s \in R \} .$$
   (a) Prove that $Z(R)$ is a subring of R.
   (b) Is $Z(R)$ always, sometimes, or never a two-sided ideal of R? Prove your answer.
   (c) Now suppose that R is a division ring. Prove that $Z(R)$ is a field.

8. (a) Let R be an integral domain, $p \in R$. Prove that if p is prime, then p is irreducible.
   (b) Now let R be a UFD. Prove that if $p \in R$ is irreducible, then p is prime.
   (c) Prove that $\mathbb{Z}[\sqrt{-5}]$ is not a UFD.

9. (a) State carefully the division algorithm for $\mathbb{R}[x]$, where $\mathbb{R}$ is the real numbers.
   (b) Using the division algorithm, prove directly (without citing theorems about Euclidean domains) that $\mathbb{R}[x]$ is a principal ideal domain.
   (c) Find all maximal ideals of $\mathbb{R}[x]$ that contain $x^8 - 1$.
   (d) Find a polynomial that generates the ideal generated by $x^{12} - 1$ and $x^8 - 1$ in $\mathbb{R}[x]$.

10. Let A and B be 2×2 matrices with entries from the real numbers. Label each of the following statements as true or false. Justify each answer with a proof or a counterexample.
   (a) If λ is an eigenvalue of both A and B, then λ is also an eigenvalue of AB.
   (b) If $\mathbf{v}$ is an eigenvector of both A and B, then $\mathbf{v}$ is also an eigenvector of AB.
   (c) If 0 is an eigenvalue of AB, then 0 is also an eigenvalue of BA.
   (d) If λ is a nonzero eigenvalue of AB, then λ is also an eigenvalue of BA.