Fall 2024 – Algebra Comprehensive Exam Name: ____________

Choose six problems total, including at least two from Part I and two from Part II. Enter the numbers of the problems you want graded here:

Problems							Total
Scores							

Part I: Groups (Choose at least two.)

1. Let $(G, *)$ be a group with operation $*$ and identity e. Let a be a fixed element of G. Define a binary operation $\triangle$ on G as $g \triangle h = g * a * h$ for all $g, h \in G$.
   (a) Prove that $(G, \triangle)$ is a group. In particular, make sure to identify the group identity under $\triangle$.
   (b) Show that $(G, *) \simeq (G, \triangle)$ by exhibiting a group isomorphism.

2. (a) Determine all finite groups that have no nontrivial proper subgroups. Justify your answer.
   (b) Determine all finite groups that have <u>exactly</u> one nontrivial proper subgroup. Justify your answer.
   (c) Give an example of a finite group G such that the number of subgroups of G is strictly greater than $|G|$, the order of G.

3. (a) Show that a group of order 28 must contain a normal subgroup of order 7.
   (b) Show that a group of order 28 must contain a normal subgroup of order 14.
   (c) Give an example of a group of order 28 that does not contain a normal subgroup of order 4. Justify your answer.
   (d) How many isomorphism classes of abelian groups of order 700 are there? For each one give its invariant factor decomposition.

4. (a) Let G be a finite group with center $Z(G)$. Prove that if $G/Z(G)$ is cyclic, then G is abelian.
   (b) Let p be a prime number, and let G be a finite <u>p-group</u>; that is, a group of order p^n for some integer n.
      i. Prove that $Z(G)$ is non-trivial.
      ii. Prove that if $|G| = p^2$, then G is abelian.

5. Show that if G is a group of order 135, then $|Z(G)| > 1$.

Part II: Rings and Linear Algebra (Choose at least two.)

6. Let R be an integral domain.
   (a) Prove: If R is finite, then R is a field.
   (b) Prove: If I is a prime ideal of R such that $|R : I|$ is finite, then I is a maximal ideal.
   (c) Give an example to show that part (b) is false without the assumption that $|R : I|$ is finite.

7. Let $R = \mathbb{Z}\left[\sqrt{-5}\right]$, and let $I = \left(3, 2+\sqrt{-5}\right)$, the ideal of R generated by 3 and $2+\sqrt{-5}$.
   (a) Determine the units of R.
   (b) Show that 3 is irreducible in R.
   (c) Show that I is not a principal ideal.
   (d) Determine a set of coset representatives for R/I.

8. Let R be a commutative ring with identity. An element $x \in R$ is <u>nilpotent</u> if $x^n = 0$ for some positive integer n.
   (a) Prove that the set $N(R)$ of nilpotent elements of R is an ideal of R. (You may use the Binomial Theorem without proof.)
   (b) Prove that if $x \in R$ is nilpotent, then x is in every prime ideal of R.
   (c) Determine the nilpotent elements of the ring $\mathbb{Z}/72\mathbb{Z}$.

9. Let $n \geq 2$ and consider the ring $\mathbb{M}_n(\mathbb{R})$ of all $n \times n$ matrices with real entries.
   (a) Exhibit a proper nontrivial left ideal of $\mathbb{M}_n(\mathbb{R})$.
   (b) Prove that $\mathbb{M}_n(\mathbb{R})$ has no proper nontrivial two-sided ideals.

10. Let n be a positive integer. Recall the standard inner product $\cdot$ on $\mathbb{R}^n$, given as follows: If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$, then $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i$. A set of vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ in $\mathbb{R}^n$ is <u>orthonormal</u> if $\mathbf{u}_1 \cdot \mathbf{u}_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$.
    (a) For fixed $\mathbf{y} \in \mathbb{R}^n$, prove that $\mathbf{x} \mapsto \mathbf{x} \cdot \mathbf{y}$ gives a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}$.
    (b) Prove that an orthonormal set in $\mathbb{R}^n$ is linearly independent.
    (c) If m is another positive integer and M is an $m \times n$ matrix with orthonormal columns, prove that $(M\mathbf{x})\cdot(M\mathbf{y}) = \mathbf{x}\cdot\mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.