Spring 2023 – Algebra Comprehensive Exam Name: ________________

Choose six problems total, including at least two from Part I and two from Part II. Enter the numbers of the problems you want graded here:

Problems							Total
Scores							

Part I: Groups (Choose at least two.)

1. Let G be a finite group with identity element e.
   (a) Prove that if $a^2 = e$ for all $a \in G$, then G is abelian.
   (b) Give an example of a nonabelian group G such that $a^4 = e$ for all $a \in G$.
   (c) Prove that if $|G| > 1$ and $a^4 = e$ for all $a \in G$, then $|Z(G)| > 1$.

2. (a) Show that if $\sigma \in S_n$ has odd order, then $\sigma \in A_n$.
   (b) Show that if a subgroup H of S_n contains an odd permutation, then $|H|$ is even and exactly half the permutations of H are odd permutations.

3. (a) Find the conjugacy classes of D_8, the dihedral group of order 8 given by $\langle r, s \mid r^4 = s^2 = e, rs = sr^{-1} \rangle$.
   (b) Let G be a nonabelian finite group with center $Z(G)$. Show that if $[G : Z(G)] = n$, then every conjugacy class of G has *strictly fewer* than n elements.
   (c) Let G be a group of order p^m where p is a prime. Prove that if $H \trianglelefteq G$ with $|H| = p$, then $H \leq Z(G)$.

4. (a) Let p and q be odd prime numbers with $p < q$ such that $p \nmid (q-1)$. Prove that any group of order pq is cyclic.
   (b) How many isomorphism classes of abelian groups of order 200 are there? For each one, give its invariant factor decomposition.

5. A subgroup H of a group G is *characteristic* if every automorphism of G maps H to H.
   (a) Prove that if H is a characteristic subgroup of G, then it is a normal subgroup of G.
   (b) Give an example of a group G with a normal subgroup H that is not characteristic.
   (c) Prove that every subgroup of a cyclic group is characteristic.

Part II: Rings and Linear Algebra (Choose at least two.)

6. Let R be a commutative ring with identity.

   (a) Let P be a prime ideal of R. Let I and J be ideals of R. Prove that if $I \cap J \subseteq P$, then $I \subseteq P$ or $J \subseteq P$.

   (b) Let M be a maximal ideal of R. Prove that R/M is a field.

   (c) Let I and J be ideals of R with $I+J = R$. Prove that $R/(I \cap J) \simeq R/I \times R/J$.

7. Prove the following statements.

   (a) Every Euclidean domain is a PID.

   (b) Every irreducible element of a PID is a prime element.

   (c) If r is an irreducible element of a PID, then (r) is a maximal ideal.

8. Let R and S be commutative rings with identity. Label each of the following statements as true or false. If true, give a proof. If false, give a counterexample.

   (a) Every ideal of $R \times S$ is of the form $I \times J$ where I is an ideal of R and J is an ideal of S.

   (b) If M is a prime ideal of R and N is a prime ideal of S, then $M \times N$ is a prime ideal of $R \times S$.

9. An element e of a ring R is called *idempotent* if $e^2 = e$. Let R be a commutative ring with identity, and suppose that $e \in R$ is an idempotent different from 0 and 1.

   (a) Prove that $1 - e$ is also idempotent.

   (b) Prove that R is not an integral domain.

   (c) Prove that the ideals (e) and $(1 - e)$ are rings with identity.

   (d) Prove that every element of R can be written as the sum of an element of (e) and an element of $(1 - e)$ in a unique way.

10. Let $M_3(\mathbb{R})$ be the set of all 3×3 matrices with entries from the real numbers $\mathbb{R}$. $M_3(\mathbb{R})$ is a vector space over $\mathbb{R}$ under the usual addition and scalar multiplication. Below are listed five subsets of $M_3(\mathbb{R})$. For each one, determine whether or not the set is a subspace of $M_3(\mathbb{R})$. Justify your answers.

    (a) The matrices with 0 determinant.

    (b) The symmetric matrices.

    (c) The invertible matrices.

    (d) The matrices having 1 as an eigenvalue.

    (e) The matrices having $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ as an eigenvector.