Fall 2023 – Algebra Comprehensive Exam Name: ________________

Choose six problems total, including at least two from Part I and two from Part II. Enter the numbers of the problems you want graded here:

Problems							Total
Scores							

Part I: Groups (Choose at least two.)

1. Let G be a group. For $g \in G$, let $|g|$ be the order of g. Suppose $x, y \in G$ with $|x| = 2$ and $|y| = 3$.
   (a) Prove that if x and y commute, then $|xy| = 6$.
   (b) Give an example of G, x, and y that satisfy the initially stated conditions (but x and y do not commute) such that $|xy| = 3$.
   (c) Give an example of G, x, and y that satisfy the initially stated conditions (but x and y do not commute) such that $|xy| = 4$.

2. (a) Find the centralizer of (134) in S_5.
   (b) Find the normalizer of $\{1, r^2 s\}$ in D_8, the dihedral group with 8 elements.
   (c) Let G be a nonabelian finite group with center $Z(G)$. Show that if $[G : Z(G)] = n$, then every conjugacy class of G has strictly fewer than n elements.

3. Let G be a group. Define the commutator subgroup of G to be the subgroup G' generated by all elements of the form $a^{-1}b^{-1}ab$, where $a, b \in G$.
   (a) Prove each of the following statements:
      i. G' is a normal subgroup of G.
      ii. The quotient group G/G' is abelian.
      iii. If $f : G \to H$ is a group homomorphism and H is abelian, then $G' \subseteq \ker f$.
   (b) Give an explicit description of G' for $G = D_8$, the dihedral group of order 8.

4. (a) Prove that any group of order 105 has a subgroup of order 35.
   (b) Describe all isomorphism classes of abelian groups of order 600.

5. For a group G, let $\mathrm{Aut}(G)$ be the group of automorphisms of G and let $Z(G)$ be the center of G.
   (a) Prove that $G/Z(G)$ is isomorphic to a subgroup of $\mathrm{Aut}(G)$.
   (b) For $G = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, determine $\mathrm{Aut}(G)$. (In other words, give a well-known group to which $\mathrm{Aut}(G)$ is isomorphic.)

Part II: Rings and Linear Algebra (Choose at least two.)

6. Let R and S be commutative rings with identity and $\phi : R \to S$ be a ring homomorphism satisfying $\phi(1_R) = 1_S$. For each statement below, prove it or give a counterexample.
   (a) If R is an integral domain, then so is $\phi(R)$.
   (b) If S is an integral domain, then $\ker\phi$ is a prime ideal.
   (c) If $P \subset S$ is a prime ideal, then $\phi^{-1}(P) = \{r \in R \mid \phi(r) \in P\}$ is a prime ideal of R.
   (d) If $M \subset S$ is a maximal ideal, then $\phi^{-1}(M)$ is a maximal ideal of R.

7. Let R be a commutative ring with identity $1_R \neq 0_R$.
   (a) Prove that if R is a principal ideal domain, then every nonzero prime ideal of R is maximal.
   (b) Prove that if $R[x]$ is a principal ideal domain, then R is a field.
   (c) Prove that if every proper ideal of R is prime, then R is a field. (*Hint*: First, show that R must be an integral domain. Then for $0 \neq r \in R$, consider the principal ideal generated by r^2.)

8. Let $R = \mathbb{Z}\left[\sqrt{-7}\right]$.
   (a) Show that 2, $\sqrt{-7}$, and $1 + \sqrt{-7}$ are irreducibles in R.
   (b) Prove that R is not a unique factorization domain.

9. (a) Let $V = \mathbb{R}[x]$.
      i. Consider the linear maps $S, T : V \to V$ defined by $S(p(x)) = p(x^2)$ and $T(p(x)) = x^2 p(x)$. Do S and T commute?
      ii. Consider the linear maps $L, M : V \to V$ defined by $L(p(x)) = \int_0^x p(t)\,dt$ and $M(p(x)) = 2p(x)$. Do S and T commute?
   (b) Show that if $A \in M_n(\mathbb{R})$ is a matrix such that $AB = BA$ for every matrix $B \in M_n(\mathbb{R})$, then $A = cI_n$ for some $c \in \mathbb{R}$.

10. (a) Let $A \in M_n(\mathbb{R})$ be a diagonalizable $n \times n$ matrix, and suppose $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A, counted with multiplicity. Fix $k \in \mathbb{N}$. Show that the eigenvalues of A^k are exactly $\lambda_1^k, \lambda_2^k, \ldots \lambda_n^k$.
    (b) Find the 2×2 matrix A that represents reflection over the line $y = 2x$ in $\mathbb{R}^2$. Then, compute A^{101}.