**Spring 2022 – Algebra Comprehensive Exam** Name: ____________

Choose six problems total, including at least two from Part I and two from Part II. Enter the numbers of the problems you want graded here:

| Problems | | | | | | | Total |
|---|---|---|---|---|---|---|---|
| Scores | | | | | | | |

**Part I: Groups** (Choose at least two.)

1. Let $G$ be a group and $H$ and $K$ be subgroups of $G$. Define $HK = \{hk \mid h \in H, k \in K\}$.
   (a) Prove that if $H$ is normal, then $HK$ is a subgroup of $G$.
   (b) Prove that if $H$ and $K$ are both normal, then $HK$ is normal.
   (c) Give an example for which neither $H$ nor $K$ is normal and $HK$ is not a subgroup of $G$.
2. For a group $G$, let $Z(G)$ be the center of $G$ and let $\text{Aut}(G)$ be the group of automorphisms of $G$.
   (a) Prove that if $G/Z(G)$ is cyclic, then $G$ is abelian.
   (b) Prove that $G/Z(G)$ is isomorphic to a subgroup of $\text{Aut}(G)$.
3. (a) Find the conjugacy classes in $A_4$. Use these to find all proper normal subgroups of $A_4$.
   (b) Show that $A_4$ is the only subgroup of $S_4$ of order 12.
   (c) Let $G$ be a nonabelian finite group with center $Z(G)$. Show that if $[G : Z(G)] = n$, then every conjugacy class of $G$ has strictly fewer than $n$ elements.
4. (a) Show that any group of order 132 is not simple.
   (b) Explain why $D_{12}$, the dihedral group of order 12, is not simple and does not have a normal Sylow 2-subgroup.
   (c) Show that there is a group of order 132 that does not have a normal Sylow 2-subgroup.
5. (a) Show that a group of order 700 has a normal subgroup of order 175.
   (b) How many isomorphism classes of abelian groups of order 700 are there? For each one give both its invariant factor decomposition and its elementary divisor decomposition.

**Part II: Rings and Linear Algebra** (Choose at least two.)

6. Let $R$ be a commutative ring with identity. An element $e \in R$ is called idempotent if $e^2 = e$. An idempotent element $e$ is called trivial if $e = 0$ or $e = 1$; otherwise it is called nontrivial. An element $r \in R$ is called nilpotent if $r^n = 0$ for some positive integer $n$.

   (a) Prove that if $e$ is idempotent, then so is $1 - e$.

   (b) Prove that a nontrivial idempotent element cannot be a unit.

   (c) Prove that if $R$ has a unique maximal ideal, then $R$ has no nontrivial idempotent elements.

   (d) Identify the units, zero divisors, idempotent elements, and nilpotent elements in the ring $\mathbb{Z}/12\mathbb{Z}$.

7. Let $R$ and $S$ be commutative rings with identity. Let $\phi : R \to S$ be a nontrivial homomorphism. For each statement below, either prove it or give a counterexample.

   (a) $\phi(R)$, the image of $\phi$, is a subring of $S$.

   (b) If $I$ is an ideal of $R$, then $\phi(I)$ is an ideal of $S$.

   (c) If $\ker \phi$ is a prime ideal of $R$, then $\phi(R)$ is an integral domain.

   (d) If $R$ is a field, then $\phi$ is injective.

8. Let $R$ be an integral domain.

   (a) Prove that if every proper ideal of $R$ is prime, then $R$ is a field. (Hint: For $0 \neq r \in R$, consider the ideal generated by $r^2$.)

   (b) Prove that if $R$ is a principal ideal domain, then every nonzero prime ideal of $R$ is maximal.

   (c) Prove that if $R$ is a unique factorization domain, then every nonzero prime ideal of $R$ contains a nonzero principal ideal that is prime.

9. (a) Show that $R = \mathbb{Z}\left[\sqrt{-2}\right]$ is a Euclidean domain under the usual norm, $N\left(x + y\sqrt{-2}\right) = x^2 + 2y^2$. In other words, show that given $a, b \in R$ with $b \neq 0$, there exist $q, r \in R$ such that $a = bq + r$ and $N(r) < N(b)$. (You may use without proof the fact that $N(ab) = N(a)N(b)$.)

   (b) The result in (a) is not true if $-2$ is replaced by $-3$. Circle the line of your work for (a) that would fail if we replaced $-2$ by $-3$.

   (c) Show carefully that $3 + \sqrt{2}$ is irreducible in $R = \mathbb{Z}\left[\sqrt{2}\right]$.

10. Construct a $3 \times 3$ matrix having an eigenvalue 1 with corresponding eigenvector $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, an eigenvalue of $-1$ with corresponding eigenvector $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and an eigenvalue of 2 with corresponding eigenvector $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Is this matrix unique, or could there be others with this property? Justify your answer.