**Fall 2022 – Algebra Comprehensive Exam** Name: ________________

Choose six problems total, including at least two from Part I and two from Part II. Enter the numbers of the problems you want graded here:

| Problems | | | | | | | Total |
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| Scores | | | | | | | |

**Part I: Groups** (Choose at least two.)

1. For each of the following statements, either prove it or give a counterexample.
   (a) If $G$ and $G'$ are groups, then any subgroup of $G \times G'$ is of the form $H \times H'$, where $H$ is a subgroup of $G$ and $H'$ is a subgroup of $G'$.
   (b) If $H$ is a subgroup of a group $G$ with $|G : H| = 2$, then $H$ is normal in $G$.
   (c) If every proper subgroup of a group $G$ is cyclic, then $G$ is cyclic.
   (d) If a group $G$ is cyclic, then all of its subgroups are cyclic.

2. Let $G$ be a group, and let $\mathrm{Bij}(G)$ be the group of bijections of $G$ to itself (i.e., the group of permutations of $G$). For each $g \in G$, define $L_g : G \to G$ by $L_g(x) = gx$ for every $x \in G$.
   (a) Show that $\Phi : g \mapsto L_g$ is a homomorphism from $G$ into $\mathrm{Bij}(G)$.
   (b) For each of the following, either find an example of a group $G$ where the property is satisfied, or explain why there are no such examples:
      i. $\Phi$ is not one-to-one;
      ii. $\Phi$ is not onto.
   (c) For each $g \in G$, define $\gamma_g : G \to G$ by $\gamma_g(x) = gxg^{-1}$ for all $x \in G$.
      i. Show that $\Gamma : g \mapsto \gamma_g$ is a homomorphism from $G$ into $\mathrm{Bij}(G)$.
      ii. Use the First Isomorphism Theorem with $\Gamma$ to produce an isomorphism of a quotient of $G$ with a subgroup of $\mathrm{Bij}(G)$.

3. Let $G$ be a group acting on a set $A$.
   (a) Prove that for any $a \in A$, the orbit of $a$ has order $|G : G_a|$, where $G_a$ is the stabilizer of $a$ in $G$.
   (b) Use this to prove that for any $g \in G$, the order of the conjugacy class of $g$ is $|G : C_G(g)|$, where $C_G(g)$ is the centralizer of $g$ in $G$.
   (c) Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 4 & 1 & 2 & 3 & 7 & 6 \end{pmatrix} \in S_7$. Determine the sizes of the conjugacy class of $\sigma$ and the centralizer $C_{S_7}(\sigma)$.

4. (a) Show that a group of order 380 contains a normal subgroup of order 95.
   (b) List all the isomorphism classes of abelian groups of order 360 (not 380). For each one, give its invariant factor decomposition.

5. Let $G$ be a finite group, let $p$ be a prime dividing $|G|$, and let $H$ be the intersection of all the Sylow $p$-subgroups of $G$.
   (a) Show that $H$ is a normal subgroup of $G$ of order $p^n$ for some integer $n \geq 0$.
   (b) Prove that if $K$ is any normal subgroup of $G$ of order $p^m$ for some integer $m \geq 0$, then $K \subseteq H$.

**Part II: Rings and Linear Algebra** (Choose at least two.)

6. Let $R = M_2(\mathbb{Z})$ be the ring of $2 \times 2$ matrices with integer entries.
   (a) For any $n \in \mathbb{Z}$, prove that the set
   $$I = \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \mid x, y, z, w \text{ are all divisible by } n \right\}$$
   is a (two-sided) ideal of $R$.
   (b) Prove that every two-sided ideal of $R$ is of this form for some integer $n$.
   (c) Give an example of a left ideal of $R$ that is not of this form.

7. Let $R$ be a commutative ring with identity. Define the Jacobson radical $\operatorname{Jac} R$ of $R$ to be the intersection of all of the maximal ideals of $R$.
   (a) Find $\operatorname{Jac}(\mathbb{Z}/600\mathbb{Z})$.
   (b) Prove that if $x \in R$ is nilpotent, that is, if $x^m = 0$ for some positive integer $m$, then $x \in \operatorname{Jac} R$.
   (c) Prove that if $r \in \operatorname{Jac} R$, then $1 + r$ is a unit of $R$.

8. Let $R = \mathbb{Z}[\sqrt{-5}]$.
   (a) Prove that $1 + \sqrt{-5}$ is irreducible but not prime in $R$.
   (b) Prove that the ideal $(2, 1 + \sqrt{-5}) \subseteq R$ is not principal but that $I^2$ is principal.

9. (a) Let $R$ be a commutative ring with identity. Prove that $R$ is a field if and only if the only ideals of $R$ are $\{0\}$ and $R$.
   (b) Give an example of a ring that has exactly three ideals.
   (c) Let $R$ be a UFD. Prove that if $a \in R$ is irreducible, then a is a prime element of $R$.

10. Let $T$ be an $n \times n$ matrix over the real numbers with the property that $T^2 = T$.
    (a) Show that for each vector $\mathbf{v} \in \mathbb{R}^n$, there is a vector $\mathbf{a}$ in the column space of $T$ and a vector $\mathbf{b}$ in the null space of $T$ such that $\mathbf{v} = \mathbf{a} + \mathbf{b}$.
    (b) Prove that the vectors $\mathbf{a}$ and $\mathbf{b}$ in part (a) are unique.
    (c) Show that $T$ is diagonalizable, and give a diagonal matrix similar to $T$.