Logical Reasoning

Rules of Logical Inference

Deductive logical reasoning is the process of deriving propositional formulas from existing one that are assumed `true`. Moreover, these derived formulas must also be `true`, assuming the truth of the ones from which they are derived. Each derived formula is obtained with the help of an **inference rule**. Each inference rule has two parts: the assumption set, and the conclusion set. The **assumption set** is a set of propositional formulas that are all assumed `true`, where as the **conclusion set** is a set of propositional formulas that must also evaluate to `true`, assuming each formula in the assumption set are true. The following is a list of the most common and useful inference rules for propositional logic.

Simplification

Assume $p \wedge q$

Conclude p, q

Conjunction

Assume p, q

Conclude $p \wedge q$

OR Inference

Assume $p \vee q$, $\neg p$

Conclude q

OR Inference

Assume $p \vee q$, $\neg q$

Conclude p

General OR Inference

Assume $p_1 \vee p_2 \vee \cdots \vee p_k$, $\neg p_1, \ldots, \neg p_{i-1}, \neg p_{i+1}, \ldots, \neg p_k$

Conclude p_i

Addition

Assume p

Conclude $p \vee q$

Addition

Assume p

Conclude $q \vee p$

Conditional Inference

Assume $p \rightarrow q$, p

Conclude q

Contrapositive Inference

Assume $p \rightarrow q$, $\neg q$

Conclude $\neg p$

XOR Inference

Assume $p \oplus q$, $\neg p$

Conclude q

XOR Inference

Assume $p \oplus q$, $\neg q$

Conclude p

XOR Inference

Assume $p \oplus q$, p

Conclude $\neg q$

XOR Inference

Assume $p \oplus q$, q

Conclude $\neg p$

Equivalence

Assume $p \leftrightarrow q$, p

Conclude q

Equivalence

Assume $p \leftrightarrow q$, q

Conclude p

Equivalence

Assume $p \leftrightarrow q$, $\neg p$

Conclude $\neg q$

Equivalence

Assume $p \leftrightarrow q$, $\neg q$

Conclude $\neg p$

Example 1. What additional information, if any, can be inferred from each of the following statements?

- John will attend only if Mary attends. Mary is not attending.
- Mary or John will attend the party. Mary is attending the party.
- You will have soup or salad, but not both. OK, you're not having the soup.
- If it rains then the sidewalk gets wet. The sidewalk got wet.

Example 1 Solution.

In addition to the above inference rules, one may also use the following identities that were stated in the Propositional Logic lecture. For each identity, if one side of the identity has been assumed or established as true, then the other side must also evaluate to `true`.

Equivalence	**Name**
$p \wedge 1 = p$ $p \vee 0 = p$	Identity
$p \vee 1 = 1$ $p \wedge 0 = 0$	Domination
$p \vee p = p$ $p \wedge p = p$	Idempotency
$\neg(\neg p) = p$	Double negation
$p \vee q = q \vee p$ $p \wedge q = q \wedge p$	Commutativity
$(p \vee q) \vee r = p \vee (q \vee r)$ $(p \wedge q) \wedge r = p \wedge (q \wedge r)$	Associativity
$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$	Distributivity
$\neg(p \vee q) = \neg p \wedge \neg q$ $\neg(p \wedge q) = \neg p \vee \neg q$	De Morgan

Example 2. Either rain or the sprinklers being turned on is sufficient for the side walk to get wet. But the sidewalk is dry. Show how De Morgan's rule is used to conclude that the sprinklers are turned off.

Example 2 Solution.

Suppose A is a set of propositional formulas, q is a single propositional formula, and V is the set of Boolean variables one which these formulas depend. Then we say that q is a **logical consequence** of A iff, for any truth assignment to the variables in V that makes every formula in A evaluate to `true`, the assignment also makes q evaluate to `true`. Moreover, an inference rule is said to be **valid** iff the conclusion C is always a logical consequence of the set of assumptions $\mathcal{A} = \{A_1, \ldots, A_k\}$. Finally, we say that a set of inference rules is **sound** iff each inference rule is valid.

There is a way to tell if an inference rule is valid. If $A_1, \ldots, A_k$ are the assumptions, and C is the conclusion, then the formula

$$A_1 \wedge A_2 \wedge \cdots \wedge A_k \rightarrow C$$

must be a tautology. In other words, make a truth table for $A_1, \ldots, A_k$, and C, and verify that the C column has a 1 whenever each of the A_i columns, $i = 1, \ldots, k$ has a 1.

Example 3. Verify that

$$(\neg Q \wedge (P \to Q)) \to \neg P$$

is a tautology. Hence, it is not valid to conclude $\neg P$ if one knows that $\neg Q$ is true, as well as $P \to Q$.

Example 3 Solution.

Logical Derivations

Given a set of propositonal formulas $A = \{p_1, \ldots, p_k\}$ that are assumed true, a **logical derivation of** q **assuming** A is a sequence of propositional formulas $f_1, f_2, \ldots, f_n$ where

1. $f_i = p_i$, $i = 1, \ldots, k$.
2. $q = f_n$ and
3. for all $i > k$, either f_i follows from a valid inference rule or logical identity involving one or more formulas in the set $\{f_1, \ldots, f_{i-1}\}$.

A logical derivation provides a **direct proof** that formula q is true whenever the formulas in A are assumed true.

Example 4. Let A denote the set of propositonal formulas listed below. Provide a logical derivation of $S \vee Q$ assuming formula in A is true.

1. $(P \vee Q) \rightarrow R$
2. $\neg Q \rightarrow S$
3. $\neg R$

Example 4 Solution.

Contradictions

A logical derivation is said to be **contradictory** iff there are two formulas, f_i and f_j, in the derivation for which one is the negation of the other. In this case we say that a contradiction has been derived.

Let $A = \{p_1, \ldots, p_k\}$ be a set of propositional formulas and V the set of variables on which the formulas in A depend. Then we say that A is a **consistent** set of formulas if iff truth values may be assigned to the variables V in such a way that all formulas in A evaluate to `true`. If no such truth values exists, then A is called an **inconsistent** set of formulas. In other words, A is inconsistent iff

$$p_1 \wedge p_2 \wedge \cdots \wedge p_k$$

is a fallacy.

Theorem 1. If a logical derivation is contradictory, then the assumption set A is **inconsistent**.

We prove Theorem 1 in a later lecture once we have studied the necessary proof techniques.

Example 5. Verify that $p \wedge \neg r$, $\neg q \vee r$, and $p \to q$ is an inconsistent set of formulas by making a truth table and verifying that their conjunction is a fallacy.

Example 6. Repeat Example 5, but now use Theorem 1 to demonstrate inconsistency by providing a contradictory derivation that has assumption set equal to

$$A = \{p \wedge \neg r, \neg q \vee r, p \to q\}.$$

Proof by contradiction

A set of inference rules is said to be **complete** iff for every set of assumptions A and every formula q that is a logical consequence of A, q can be logicaly derived from A via a direct step-by-step proof. Otherwise, the set of rules is said to be **incomplete**. It turns out that the inference rules that we've introduced in this lecture are incomplete. In other words, there are times when a formula q is a logical consequence of a set of formulas A, but there is no way to derive q using A as the assumption set along with all the provided rules of inference. When this happens, we need to use other means to establish that q is a logical consequence of A. Of course, we may always use a truth table, but sometimes there are too many variables for this to be practical.

In what follows, let A be a consistent set of formulas. Another strategy for showing that q is a logical consequence of A is to make $A \cup \{\neg q\}$ the assumption set and derive a contradiction. This technique is called **proof by contradiction**. By Theorem 1, this would imply that $A \cup \{\neg q\}$ is an inconsistent set. But we have assumed that A is consistent, meaning that there is at least one variable assignment α that makes A true. Moreover, since $A \cup \{\neg q\}$ is inconsistent, it means that α makes $\neg q$ `false`, which means that it makes q `true`. Therefore, q evaluates to `true` whenever an assignment α makes each formula in A evaluate to `true`, and so we have proved that q is a logical consequence of p.

Note: if q is a logical consequence of A, then it does *not* necessarily mean that there exists a logical derivation of q whose assumption set is A (since the set of inference rules may be incomplete). However, the converse *is* true: if assumption set A derives q then q is a logical consequence of A.

Example 7. Let A denote the assumption set from Example 4. Show that $\neg(S \vee Q)$ is a logical consequence of A by assuming $A \cup \{\neg(S \vee Q)\}$ and deriving a contradiction.

1. $(P \vee Q) \rightarrow R$
2. $\neg Q \rightarrow S$
3. $\neg R$
4. $\neg(S \vee Q)$

Proof by Cases

Suppose we have a consistent set of formulas A, and we wish to show that q is a logical consequence of A. Suppose further that we have failed to directly derive q from A. Aside from a proof by contradiction, another technique that may be used is called **proof by cases**. Using this technique, we may add one or more additiional assumptions to A. For simplicity, in this lecture we assume that exactly one additional formula f is added to A and that one of two outcomes occur after adding f to A.

Case 1. by adding f to A we are now able to derive q. In this case we must now start over by now adding $\neg f$ to A and either deriving q or deriving a contradiction. If we derive q again, then it means that q is a logical consequence of A, since every assignment α that satisfies A must either satisfy f or $\neg f$ in which case, based on either of the two derivations, α will satisfy q. On the other hand, should we derive a contradiction, then f is a logical consequence of A, which makes q a logical consequence of A (explain why this is the case).

Case 2. by adding f to A we are now able to derive a contradiction. In this case we have learned that $\neg f$ is a logical consequence of A, and so $\neg f$ may be added as an additional assumption without jeapordizing consistency. We then start a new derivation and see if we are able to derive q. If yes, we are done. If no, we may repeat the proof-by-cases technique again.

Example 8a. Consider the following consistent set $\mathcal{A}$ of propositional formulas. Show that by adding formula G to this set, we are able to derive a contradiction.

1. $D \oplus G$
2. $G \rightarrow (\neg E \wedge D)$
3. $A \vee F \vee \neg G$
4. $A \leftrightarrow G$
5. $\neg C \rightarrow (A \vee B)$
6. $C \rightarrow (A \wedge \neg D)$
7. $A \vee C \vee E$
8. $B \oplus F$

Example 8a. Consider the following consistent set $\mathcal{A}$ of propositional formulas. Show that by adding formula $\neg G$ to this set, we are able to derive the formula

$$\mathcal{F} = \neg A \wedge B \wedge \neg C \wedge D \wedge E.$$

Conclude that $\mathcal{F}$ is a logical consequence of $\mathcal{A}$.

1. $D \oplus G$
2. $G \rightarrow (\neg E \wedge D)$
3. $A \vee F \vee \neg G$
4. $A \leftrightarrow G$
5. $\neg C \rightarrow (A \vee B)$
6. $C \rightarrow (A \wedge \neg D)$
7. $A \vee C \vee E$
8. $B \oplus F$

Example 9a. Knights always tell the truth while Knaves always lie. Assume Alice is either a Knight or a Knave. Assume the same about Bob. Alice says "Bob and I are both knaves". What can you conclude from this statement?

Example 9b. Use the argument from Example 9a as a guide to formally proving that the Example-9a conclusions are a logical consequence of the Example-9a assumptions.

Exercises

1. Consider the inference rule where one assumes p and $p \vee q$ are true, and then concludes that q must be true. Show that this inference rule is not valid.

2. Consider the inference rule where one assumes p and $p \leftrightarrow q$ are true, and then concludes that q must be true. Show that this inference rule is valid.

3. Let $A = \{(a \vee b) \to (c \vee d), \neg c, b\}$. Using any logical inferences or identities presented in this lecture, provide a logical derivation of d assuming A.

4. Let $A = \{(a \wedge b) \to (c \wedge d), \neg c, b\}$. Using any logical inferences or identities presented in this lecture, provide a logical derivation of $\neg a$ assuming A.

5. Either Paul or Quan (or both) will attend CSULB. If Paul attends, then so will Sam. But Sam will not attend if his brother Robert attends. Finally, it was learned that Sam did not attend CSULB.

   a. Define propositional variables p, q, r, and s and model each of the given facts with a propositional formula. Construct an exhaustive LDT to determine which of the four will attend CSULB.

   b. Provide a logical derivation whose assumptions are the formulas from part a, along with r (Robert is attending), and whose conclusion is $\neg p \wedge q$.

   c. Provide a logical derivation whose assumptions are the formulas from part a, along with $\neg r$ (Robert is not attending), and whose conclusion is $\neg p \wedge q$.

   d. Based on parts b and c, what can we say about Paul's, Quan's, and Robert's attendance?

6. Four friends have been identified as suspects for a burglary that police believe to be committed by only one person. They made the following statements to the police. Alice said, "Carlos did it". John said, "I did not do it". Carlos said, "Diana did it". Diana said, "Carlos lied when he said that I did it". Define propositional variables so that you can model these statements, including variables that indicate whether or not a given statement is true. The police know that exactly one of the four suspects is telling the truth. Use an exhaustive LDT to determine who should be charged with the burglary. Hint: make "Assume tc" your first unforced assumption.

7. Alice says "Bob and I are both knights". Bob says, "Alice is a knave". What can you conclude from this? Explain.

8. Alice says "Bob and I are both knaves". Bob says nothing. What can you conclude from this? Explain.

9. You approach two men, each of whom is standing next to a door. One door leads to room with a pot of gold, while the other leads to a deadly encounter with a python snake. You know that one man is a knight, while the other is a knave, but you do not know which is the knight. What one question should you ask (to either man) in order to determine the room with the gold? Note that you can ask only one of the men.

10. Abe, Ben, Carl, and Darin may or may not attend a baseball game. The following facts will determine who attends.

a. Carl will attend provided Ben attends.

b. Either Abe is not attending or Carl is not attending.

c. Either Abe or Darin is attending.

d. If Darin attends then Ben will attend but Carl will not attend (Carl does not like Darin).

- **a.** Using propositional atoms a, b, c, and d, write propositional formulas that express each of the above facts.
- **b.** Provide an exhaustive LDT to determine who will attend the game.

11. Translate the following clues about which of three friends (Al, Ben, and Cris who represent the universe of discourse) are attending a movie.

    - **a.** Cris is attending if either Al is attending or Ben is not attending. If Cris is attending then everyone is attending. If Al is attending, then there is someone who is not attending. There exists someone who is attending. (10 points)
    - **b.** Provide an exhaustive LDT of who is (and is not) attending the movie.

Exercise Answers and Hints

1. Show that $(p \wedge (p \vee q)) \rightarrow q$ is not a tautology.

2. Show that $(p \wedge (p \leftrightarrow q)) \rightarrow q$ is a tautology.

3. The following is a logical derivation tree.

   1. $(a \vee b) \rightarrow (c \vee d)$
   2. b
   3. $\neg c$
   4. $a \vee b$ (2)
   5. $c \vee d$ (1,4)
   6. d (3,5)

4. The following is a logical derivation tree.

   1. $(a \wedge b) \rightarrow (c \wedge d)$
   2. b
   3. $\neg c$
   4. $\neg c \vee \neg d$ (3)
   5. $\neg(c \wedge d)$ (4, De Morgan)
   6. $\neg(a \wedge b)$ (1,5)
   7. $\neg a \vee \neg b$ (6, De Morgan)
   8. $\neg a$ (2,7)

5. a. $p \vee q$, $p \rightarrow s$, $r \rightarrow \neg s$, $\neg s$

   b.
   1. $p \vee q$
   2. $p \rightarrow s$
   3. $r \rightarrow \neg s$
   4. $\neg s$
   5. r
   6. $\neg p$ (2,4, contrapos.)
   7. q (1,6, OR)
   8. $\neg p \wedge q$ (6,7,Conjunction)

   c.
   1. $p \vee q$
   2. $p \rightarrow s$
   3. $r \rightarrow \neg s$
   4. $\neg s$
   5. $\neg r$
   6. $\neg p$ (2,4, contrapos.)

7. q (1,6, OR)
8. $\neg p \wedge q$ (6,7,Conjunction)

d. Paul not attending and Quan attending are both logical consequences of the original problem statements. These statements have no bearing on whether or not Robert will attend.

6. a. **Variables:** ta (Alice is telling the truth), tc (Carlos is telling the truth), td (Diana is telling the truth), tj (John is telling the truth), ba (Alice is the burglar), bc (Carlos is the burglar), bd (Diana is the burglar), bj (John is the burglar).

   b.
   1. Exactly one of the four suspects is the burglar.
   2. Exactly one of the four suspects is telling the truth.
   3. $ta \leftrightarrow bc$
   4. $tc \leftrightarrow bd$
   5. $td \leftrightarrow \neg tc$
   6. $tj \leftrightarrow \neg bj$
   7. Assume tc
      8. $\neg ta \wedge \neg td \wedge \neg tj$ (2,7)
      9. bd (4,7)
      10. $\neg ba \wedge \neg bc \wedge \neg bj$ (1,9)
      11. $\neg bj$ (10)
      12. tj (6,11)
      13. $\neg tj$ (8)

      Contradiction: (12,13) ❒
   14. Assume $\neg tc$
      15. $\neg bd$ (4,14)
      16. td (5,14)
      17. $\neg ta \wedge \neg tc \wedge \neg tj$ (2,7)
      18. $\neg tj$ (17)
      19. $\neg\neg bj$ (6,18)
      20. bj (19, Double Negation)

      Conclusion: John is the burglar. ❒

      Note: we cheated a bit here, because the branch of an LDT should end only when all variables have been assigned (i.e. either the variable or its literal have been established as true). We leave it as an exercise to establish the truth or falsehood of the remaining variables, and check that no contradiction exists.

7. **Variables:** ta (Alice is telling the truth), tb (Bob is telling the truth), ka (Alice is a knight), kb (John is a knight).

   1. $ta \leftrightarrow ka$
   2. $tb \leftrightarrow kb$
   3. $ta \leftrightarrow (ka \wedge kb)$
   4. $tb \leftrightarrow \neg ka$

5. Assume ka
    6. ta (1,5)
    7. $(ka \wedge kb)$ (3,6)
    8. kb (7)
    9. tb (2,8)
    10. $\neg ka$ (4,9)

    Contradiction: (5,10) ❒
11. Assume $\neg ka$
    12. tb (4,11)
    13. kb (2,12)
    14. $\neg ta$ (1,11)

    Conclusion: Alice is a knave and Bob is a knight which is consistent with statements 1-4. ❒

8. **Variables:** ta (Alice is telling the truth), tb (Bob is telling the truth), ka (Alice is a knight), kb (John is a knight).

    1. $ta \leftrightarrow ka$
    2. $tb \leftrightarrow kb$
    3. $ta \leftrightarrow (\neg ka \wedge \neg kb)$
    4. Assume ta
        5. $(\neg ka \wedge \neg kb)$
        6. $\neg ka$ (5)
        7. $\neg ta$ (1,6)

        Contradiction: (4,7) ❒
    8. Assume $\neg ta$
        9. $\neg(\neg ka \wedge \neg kb)$ (3,8)
        10. $\neg\neg ka \vee \neg\neg kb$ (9, De Morgan)
        11. $\neg ka$ (1,8)
        12. $\neg\neg kb$ (10,11)
        13. kb (12, Double Negation)
        14. tb (2,13)

        Conclusion: Alice is a knave and Bob is a knight which is consistent with statements 1-3. ❒

9. Hint: ask one of the men a question that pertains to the other man.

10. **Variables:** a (Abe is attending), b (Ben is attending), c (Carl is attending), d (Darin is attending).

    1. $b \rightarrow c$
    2. $\neg a \vee \neg c$

3. $a \vee d$
4. $d \rightarrow (b \wedge \neg c)$
5. Assume d
    6. $(b \wedge \neg c)$ (4,5)
    7. b (6)
    8. c (1,7)
    9. $\neg c$ (6)

    Contradiction: (8,9) ❒
10. Assume $\neg d$
    11. a (3,10)
    12. $\neg c$ (2,11)
    13. $\neg b$ (1,12)

    Conclusion: satisfying assignment $(a = 1, b = 0, c = 0, d = 0)$, and only Abe is attending. Note: statements 1-4 are all satisfied. ❒

11. **Variables:** a (Al is attending), b (Ben is attending), c (Cris is attending).

    1. $(a \vee \neg b) \rightarrow c$
    2. $c \rightarrow (a \wedge b \wedge c)$
    3. $a \rightarrow (\neg a \vee \neg b \vee \neg c)$
    4. $a \vee b \vee c$
    5. Assume c
        6. $a \wedge b \wedge c$ (2,5)
        7. a (6)
        8. $\neg a \vee \neg b \vee \neg c$ (3,7)
        9. $\neg\neg a$ (7, Double Negation)
        10. $\neg\neg c$ (5, Double Negation)
        11. $\neg b$ (8,9,10)
        12. b (6)

        Contradiction: (11,12) ❒
    13. Assume $\neg c$
        14. $\neg(a \vee \neg b)$ (1,13)
        15. $\neg a \wedge \neg\neg b$ (14, De Morgan)
        16. $\neg a$ (15)
        17. $\neg\neg b$ (15)
        18. b (17, Double Negation)

        Conclusion: satisfying assignment $(a = 0, b = 1, c = 0)$, and only Ben is attending. Note: statements 1-4 are all satisfied. ❒