Interpolation

Introduction

Given points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$, with $x_0 < x_1 < \cdots < x_n$, the **interpolation** of these points refers to defining one or more (usually smooth) functions that together forms a function $f(x)$ that i) is defined over the interval $[x_0, x_n]$, and ii) for which $f(x_i) = y_i$ for all $i = 0, 1, \ldots, n$.

In this lecture we examine three approaches: direct-fitting polynomials, Lagrange polynomials, and **Newton's Divided Difference Method** which determines the polynomial coefficients via derivative approximations.

Direct-fitting polynomials

Given set of points $P = \{(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)\}$, the n th order **direct-fitting polynomial** $p_n(x)$ with respect to P is given by

$$p_n(x) = a_0 + a_1 x + \cdots + a_n x^n,$$

where $a_0, a_1, \ldots, a_n$ satisfy the system of equations

$$\begin{gathered} a_0 + a_1 x_0 + \cdots + a_n x_0^n = y_0 \\ a_0 + a_1 x_1 + \cdots + a_n x_1^n = y_1 \\ \vdots \\ a_0 + a_1 x_n + \cdots + a_n x_n^n = y_n \end{gathered}$$

The above system can be written in matrix form as

$$\begin{pmatrix} 1 & x_0 & \cdots & x_0^n \\ 1 & x_1 & \cdots & x_1^n \\ 1 & \vdots & \cdots & \vdots \\ 1 & x_n & \cdots & x_n^n \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}$$

Example 1. Determine the equation of the unique line that passes through the points $(1, -2)$ and $(3, 5)$.

Example 2. Determine the equation of the unique quadratic polynomial that passes through the points $(-2, 5)$, $(3, 0)$, and $(0, -3)$.

Lagrange polynomials

Given set of points $P = \{(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)\}$, the n th order **Lagrange polynomial** $p_n(x)$ with respect to P is given by

$$p_n(x) = \sum_{i=0}^{n} L_i(x) y_i,$$

where

$$L_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}.$$

Again, the derivative at any point x_i can then be approximated as $p_n'(x_i)$.

Example 3. Repeat Example 1 using a Lagrange polynomial and verify that it agrees with the Example 1 solution.

Example 4. Repeat Example 2 using a Lagrange polynomial and verify that it agrees with the Example 1 solution.

Newton's Divided Difference Method

The general form of Newton's divided-difference interpolating polynomial is

$$p_n(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + \cdots + b_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}),$$

where the b_i are recursively computed.

To define the b_i, $i = 0, \ldots, n$, we first define the quantities $f[x_i, x_{i-1}, \ldots, x_{i-j}]$, where $0 \le j \le i$.

Base Case 1. $f[x_i] = y_i$ is the y-value associated with x_i.

Base Case 2. $f[x_i, x_{i-1}] = \frac{y_i - y_{i-1}}{x_i - x_{i-1}}$ is the average rate of change from x_{i-1} to x_i. In other words, we may think of $f[x_i, x_{i-1}]$ as a first derivative approximation.

Recursive Case. Suppose $f[x_i, x_{i-1}, \ldots, x_{i-j+1}]$ and $f[x_{i-1}, x_{i-2}, \ldots, x_{i-j}]$ have both been defined, while the first represents the approximation of a $(j-1)$ th derivative over the interval $[x_i, x_{i-j+1}]$, while the second represents the approximation of a $(j-1)$ th derivative over the interval $[x_{i-1}, x_{i-j}]$. Then $f[x_i, x_{i-1}, \ldots, x_{i-j}]$ is defined as

$$f[x_i, x_{i-1}, \ldots, x_{i-j}] = \frac{f[x_i, x_{i-1}, \ldots, x_{i-j+1}] - f[x_{i-1}, x_{i-2}, \ldots, x_{i-j}]}{x_i - x_{i-j}}$$

and represents the approximation of a j th derivative over the interval $[x_i, x_{i-j}]$.

Example 5. Consider the three points $(x_0, y_0) = (-1, 2)$, $(x_1, y_1) = (2, 3)$, and $(x_2, y_2) = (3.5, 7)$, compute $f[x_0]$, $f[x_1]$, $f[x_2]$, $f[x_1, x_0]$, $f[x_2, x_1]$, and $f[x_2, x_1, x_0]$.

We are now in a position to define the b_i coefficients of Newton's divided-difference polynomial.

Theorem 1. Given points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$, and $b_i = f[x_i, \ldots, x_0]$, for all $i = 0, 1, \ldots, n$, then the divided-difference polynomial

$$p_n(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + \cdots + b_n(x - x_0)(x - x_1) \cdots (x - x_{n-1})$$

has the property that, for all $i = 0, 1, \ldots, n$, $p_n(x_i) = y_i$.

The proof Theorem 1 uses induction and is left as an exercise.

Example 6. Verify that $p_n(x_i) = y_i$, for $i = 0, 1, 2$.

Exmaple 7. Compute the divided-difference polynomial $p_2(x)$ for the data points provided in Example 1.

Exercises

1. Find the direct-fitting polynomial $p_1(x)$ with respect to the points $\{(-1, 2), (3, 6)\}$.

2. Find the direct-fitting polynomial $p_2(x)$ with respect to the points $\{(-1, 9), (1, 3), (2, 3)\}$.

3. Determine the Largrange polynomial $p_3(x)$ associated with the points

$$(0, 4), (1, 3), (4, -2), (5, 1)$$

4. An aircraft position during an emergency landing exercise on a runway was timed according to the following table.

time t	0	0.4	1.00	1.75	2.5
position x	20	71	110	161	178

Determine the fourth-degree Lagrange polynomial $p(t)$ with respect to this data.

5. Determine the divided-difference polynomial $p_1(x)$ with respect to points $(1, 4)$, $(3, 8)$.

6. Determine the divided-difference polynomial $p_2(x)$ with respect to points $(1, 4)$, $(2, -1)$, $(5, -8)$.

7. The following table provides measurements of the velocity of a body at different moments in time.

Time (s)	0	15	18	22	24
Velocity (m/s)	22	24	37	55	123

Use a 2nd-order divided-difference polynomial $p_2(x)$ to approximate the velocity at $t = 14$s.

8. Prove that if

$$p_2(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1),$$

then $p_2(x_i) = y_i$, for all $i = 0, 1, 2$.

9. Prove Theorem 1 using mathematical induction.

Exercise Hints and Answers

1. $p_1(x) = 3 + x$

2. $p_2(x) = x^2 - 3x + 5$

3. We have

$$f(x) = \sum_{i=0}^{3} L_i(x)y_i =$$

$$\frac{4(x-1)(x-4)(x-5)}{(-1)(-4)(-5)} + \frac{3(x)(x-4)(x-5)}{(1)(-3)(-4)} + \frac{-2(x)(x-1)(x-5)}{(4)(3)(-1)} + \frac{1(x)(x-1)(x-4)}{(5)(4)(1)}.$$

4. We have

$$p(t) = \sum_{i=0}^{4} L_i(t)x_i =$$

$$\frac{20(t-0.4)(t-1.00)(t-1.75)(t-2.5)}{(-0.4)(-1.00)(-1.75)(-2.5)} + \frac{71(t)(t-1.00)(t-1.75)(t-2.5)}{(0.4)(-0.6)(-1.35)(-2.1)} +$$
$$\frac{110(t)(t-0.4)(t-1.75)(t-2.5)}{(1.00)(0.6)(-0.75)(-1.5)} + \frac{161(t)(t-0.4)(t-1.00)(t-2.5)}{(1.75)(1.35)(0.75)(-0.75)} +$$
$$\frac{178(t)(t-0.4)(t-1.00)(t-1.75)}{(2.5)(2.1)(1.5)(0.75)}.$$

5. We have

$$p_1(x) = f[x_0] + f[x_1, x_0](x - x_0) = f[x_0] + f[x_1, x_0](x - 1).$$

Moreover, $f[x_0] = 4$,

$$f[x_1, x_0] = \frac{8-4}{3-1} = 2,$$

Therefore,

$$p_1(x) = 4 + 2(x - 1).$$

6. We have

$$p_2(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1) =$$
$$f[x_0] + f[x_1, x_0](x - 1) + f[x_2, x_1, x_0](x - 1)(x - 2).$$

Moreover, $f[x_0] = 4$,

$$f[x_1, x_0] = \frac{-1-4}{2-1} = -5,$$

$$f[x_2, x_1] = \frac{-8-(-1)}{5-2} = --7/3,$$

$$f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{5-1} = \frac{(-7/3) - (-5)}{5-1} = 2/3.$$

Therefore,

$$p_2(x) = 4 - 5(x - 1) + 2/3(x - 1)(x - 2).$$

7. The NDD polynomial with respect to points $(0, 22)$, $(15, 24)$, $(18, 37)$ is

$$p_2(t) = 22 + \frac{2}{15}t + \frac{189}{810}t(t - 15).$$

Moreover,

$$p_2(14) = 22 + \frac{2}{15}(14) + \frac{189}{675}(14)(14 - 15) = 19.9467.$$