Numerical Methods for Solving Equation $f(x) = 0$

Bisection Method

In the following we use the function $\mathrm{sgn}(x)$, which is defined as

$$\begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

Step 1. Find x_l and x_u for which $f(x_l)f(x_u) < 0$

Step 2. Compute $x_m = (x_l + x_u)/2$

Step 3. If $f(x_m) = 0$, then a solution has been found.

Step 4. If $\mathrm{sgn}(f(x_m)) = \mathrm{sgn}(f(x_l))$, then $x_l \leftarrow x_m$, and go to Step 2.

Step 5. If $\mathrm{sgn}(f(x_m)) = \mathrm{sgn}(f(x_u))$, then $x_u \leftarrow x_m$, and go to Step 2.

Note: during each iteration, the official estimate of the root is $x_m = (x_l + x_u)/2$.

Example 1. Given $f(x) = x^2 + x - 2$ with zero $z = 1$, if the initial lower and upper values are $x_l = 0$ and $x_u = 3$, then compute the first three values of x_m, along with the absolute i) true error, ii) relative true error, iii) approximation error, and iv) relative approximation error.

Iteration	x_l	x_u	x_m	**T.E.**	**A.E.**	**R.A.E.**
1	0	3	1.5	0.5	n/a	n/a
2	0	1.5	0.75	0.25	0.75	0.5
3	0.75	1.5	1.125	0.125	0.375	0.5

Newton-Raphson Method

Given function $f(x)$, we say that z is a **zero** of f iff $f(z) = 0$. The **Newton-Raphson method** assumes that $f(x)$ is differentiable in an interval that contains zero z. Moreover, since a differentiable function can be approximated about a point $(x, f(x))$ using the tangent line L passing through $(x, f(x))$, if x is near z, then the x-intercept of L should be near z.

Theorem 1. Let x_i be the current approximation for the zero of a differentiable function $f(x)$. Moreover, suppose $f'(x_i) \neq 0$. Then if x_{i+1} is the x-intercept of the tangent line passing through point $(x_i, f(x_i))$, then we have

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}.$$

Proof of Theorem 1. The equation of the tangent line passing through point $(x_i, f(x_i))$ is given by

$$y - f(x_i) = f'(x_i)(x - x_i).$$

Moreover, setting $y = 0$, $x = x_{i+1}$, and solving for x_{i+1} yields

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}.$$

□

Example 2. Given $f(x) = x^2 + x - 2$ with zero $z = 1$, if the initial approximation of z is $x_0 = 2$, then compute x_1 and x_2 using the Newton-Raphson method, along with the absolute i) true error, ii) relative true error, iii) approximation error, and iv) relative approximation error.

Iteration	x_i	**T.E.**	**A.E.**	**R.A.E.**
0	2	1	n/a	n/a
1	1.2	0.2	0.8	0.4
2	1.01177	0.01177	0.188235	0.156863

Secant Method

The **secant method** is based on the same idea as the Newton-Raphson method, except now we assume that f's derivative is not available to use. Instead, $f'(x_i)$ is approxomated as

$$f'(x_i) \approx \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}},$$

where x_{i-1} is the previous approximation.

This gives the new recurrence

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}.$$

Example 3. Repeat Example 2 using the Secant method, and $x_0 = 0$, $x_1 = 3$.

Iteration	x_i	**T.E.**	**A.E.**	**R.A.E.**
0	0	1	n/a	n/a
1	3	2	n/a	n/a
2	0.5	0.5	2.5	0.833333
3	0.777778	0.222222	0.277778	0.555556
4	1.04878	0.04878	0.271002	0.348310

Exercises

Note: for this assignment use *at least* six decimal places (to the right of the decimal) in your calculations of true and approximate values. Doing this will ensure your answers are sufficiently close to the ones provided.

1. Assuming an initial interval of $[1, 5]$ determine the estimate of the root to the equation $te^{-t} - 0.3$ during iteration 2 of the Bisection method.

2. For a given iteration of the Bisection method the absolute relative approximate error is defined as
$$\epsilon_a = |x_m - x'_m|/x_m,$$
where x'_m is the estimate from the previous iteration. Show that
$$\epsilon_a = |x_l - x_u|/(x_l + x_u),$$
Hint: use a proof by cases on the case of when $x_l = x'_m$, and on the case of when $x_u = x'_m$.

3. For the equation $x^2 - 4 = 0$ and starting interval $[1.7, 2.4]$, make a table whose rows represent the first four iterations of the bisection method, and whose columns are x_l, x_u, x_m, the (absolute) approximate error, relative approximate error, true error, and relative true error,

4. For the equation $te^{-t} + \frac{1}{t} - 0.35 = 0$ and starting interval $[1, 8]$, make a table whose rows represent the first three iterations of the bisection method, and whose columns are x_l, x_u, x_m, the (absolute) approximate error, and the relative approximate error,

5. For an application of the bisection method, suppose the starting interval is $[1, 3]$. Provide the worst-case absolute relative true error with respect to the approximation provided in iteration 3.

6. Repeat the previous exercise, but now provide a general formula for the absolute relative true error with respect to the approximation provided in iteration $n \geq 1$.

7. For the equation $x^2 - 4 = 0$ and starting with $x_1 = 1$ make a table whose rows represent the first four iterations of the Newton-Raphson method, and whose columns are i, x_i, the (absolute) approximate error, relative approximate error, true error, and relative true error,

8. For the equation $te^{-t} + \frac{1}{t} - 0.35 = 0$ and starting with $x_1 = 3$, make a table whose rows represent the first three iterations of the Newton-Raphson method, and whose columns are i, x_i, the (absolute) approximate error, and the relative approximate error,

9. Show that the Newton-Raphson finds the zero aftre one iteration in the case the $f(x)$ is a linear function with nonzero slope.

10. For the equation $x^2 - 4 = 0$ and starting with $x_0 = 1$, $x_1 = 3$, make a table whose rows represent the first four iterations (i.e. up to x_4 of the secant method, and whose columns are i, x_i, the (absolute) approximate error, relative approximate error, true error, and relative true error,

11. For the equation $te^{-t} + \frac{1}{t} - 0.35 = 0$ and starting with $x_0 = 3$, $x_1 = 4$, make a table whose rows represent the first three iterations of the secant method, and whose columns are i, x_i, the (absolute) approximate error, and the relative approximate error,

Exercise Hints and Answers

1. For iteration 1 we have $x_m = 3$, with $3e^{-3} - 0,3 = -0.1506388 < 0$, and so $x_u = 3$ for iteration 2. Therefore $x_m = (1+3)/2 = 2$ is the approximation during iteration 2.

2. Case 1: $x_l = x_m'$. Then we have

   $$\epsilon_a = |x_m - x_l|/(x_l + x_u)/2 = 2|(x_l + x_m)/2 - x_l|/(x_l + x_u) = (2/2)|x_m - x_l|/(x_l + x_u) = |x_l - x_u|/(x_l + x_u).$$

   A similar derivation can be performed in the case that $x_u = x_m'$.

3. We have the following table.

Iteration	x_l	x_u	x_m	**A.E.**	**R.A.E.**	**T.E.**	**R.T.E.**
1	1.7	2.4	2.05	n/a	n/a	0.05	2.5%
2	1.7	2.05	1.875	0.175	9.333%	0.125	6.25%
3	1.875	2.05	1.9625	0.0875	4.413%	0.0375	1.875%
4	1.9625	2.05	2.00625	0.04375	2.189%	0.00625	0.3125%

4. We have the following table.

Iteration	x_l	x_u	x_m	**A.E.**	**R.A.E.**
1	1	8	4.5	n/a	n/a
2	1	4.5	2.75	1.75	63.63636%
3	2.75	4.5	3.625	0.875	24.13793%

5. To maximize the relative error, the solution should be as small as possible. Although it cannot equal $x = 1$, it can be arbitrarily close to 1. In this case the approximations for the first three iterations are 2, 1.5, and 1.25. This gives an absolute, true error close to 0.25, which is also the value of the absolute relative error.

6. $1/2^{n-1}$

7. We have the following table. Note: $x = 2$ is the equation root.

i	x_i	**A.E.**	**R.A.E.**	**T.E.**	**R.T.E.**
1	1	n/a	n/a	1	100%
2	2.5	1.5	60%	0.5	25%
3	2.05	0.45	21.95%	0.05	2.5%
4	2.00061	0.04939	2.5%	0.00061	0.0305%

8. We have the following table.

i	x_i	**A.E.**	**R.A.E.**
1	3	n/a	n/a
2	3.629824	0.629824	17.35%
3	3.78	0.150	3.97%

9. Assume $f(x) = ax + b$, where $a \neq 0$. Let x_i be any number. Then, using Newton-Raphson,

   $$x_2 = x_1 - (ax_1 + b)/a = x_1 - x_1 - b/a = -b/a$$

   which is the root of the equation $ax + b$.

10. We have the following table. Note: $x = 2$ is the equation root.

i	x_i	**A.E.**	**R.A.E.**	**T.E.**	**R.T.E.**
0	1	n/a	n/a	1	100%
1	3	n/a	n/a	1	100%
2	1.75	1.25	71.43%	0.125	12.5%
3	1.947	0.197	10.1%	0.053	2.65%
4	2.0036	0.056584	2.82%	0.0036	0.18%