Sample project topics Math 456

Bear in mind: The quality of a project is directly related to its having **substantive content** and a **well-defined**, **narrow focus**. The following are intended to be **suggestive**. Most topics need a narrowing of focus. Computer graphics and experiments might help to illuminate some of the ideas and results.

One-dimensional dynamics

- Devise a *non-linear* method of paper-folding. Analyze its dynamics.
- The calculus-based technique known as **Newton's method** is one of the best-known iterative procedures for asymptotocally finding where a function is 0. Let $f(x) = x(x^2 - 4)$. Using both theory and experiment, work out Newton's method applied to f in dynamical terms—fixed points, critical points, attracting points and their basins. Are there points whose orbits fail to converge to the roots of $f(x)$?
- Explore the variant of Newton's method known as *Halley's method.* Briefly, the idea is to use second order approximation of a function f to define a dynamical system that finds the zeros of f. A good case on which to focus and contrast with Newton's method is the cubic polynomial.
- Explore Newton's method for one-variable functions of *complex* numbers. Contrast this case with that of functions of one real variable.
- Sharkovskii's Theorem tells us when, under a continuous map on $\mathbf{R}$, points with certain periods arise. A "converse" question asks whether there are maps with *exactly* the periods required by Sharkovskii's result.

  Reference: S. Elaydi, *On a converse of Sharkovskii's Theorem*
- Investigate and explain some of the basic theory of *cellular automata.*

  Reference: P. Kurka, *Topological and Symbolic Dynamics*
- The phenomenon of shadowing associated with a trajectory is crucial for the reliability of a trajectory that a computer generates. Discuss the basic issue of shadowing and examine the question of when a map has the "shadowing property."
- Morse-Smale maps are a special class of maps on the circle. Develop the elementary theory of these maps—in particular, their structural stability.

  Reference: R. Devaney, *An Introduction to Chaotic Dynamical Systems*
- The *Feigenbaum constant* is a number that arises when maps undergo period doubling. A fascinating result is that this constant is universal across all maps satisfying a certain condition on extreme values. Develop the beginning theory of the constant as it concerns one-dimensional maps—such as the logistic family. Compute and examine the bifurcation diagrams for several families of maps to "see" how the constant arises.

  Reference: K. Briggs . "A Precise Calculation of the Feigenbaum Constants
- Conduct a deeper exploration of the Schwarzian derivative. In particular, how do dynamical properties of a map with negative Schwarzian generalize to the case of a map with *eventual* negative Schwarzian.

  Reference: B. Webb, "Dynamics of Functions with an Eventual Negative Schwarzian Derivative"

- Investigate the question of computing a trajectory for a map on an interval. Discuss how the issue of complexity arises in the presence of chaos.

  Reference: C. Spandl, "Computational Complexity of Iterated Maps on the Interval"
- Develop the basic theory of *renormalization* for a map.

  Reference: R. Devaney, *An Introduction to Chaotic Dynamical Systems*
- Create a computer algorithm for computing the Lyapunov number/exponent $L(a)$ for maps in the logistic family
$$x \longrightarrow a\,x(1-x) \qquad 2 \leq a \leq 4.$$
  Graph $L(a)$ as a function of a.
- Under a map f, a point p is *non-wandering* if, for every neighborhood N of p,
$$f^k(N) \cap N \neq \emptyset \qquad \text{for some } k > 1.$$
  Let Ω_f be the set of non-wandering points under f. If $f^k(p)$ eventually returns to N, p is called *recurrent*.

  1) Show that Ω_f is closed. That is, contains all of its limit points.
  2) Show that Ω_f is *invariant* under f. That is, $f(\Omega_f) = \Omega_f$.
  3) It's clear that a recurrent point is non-wandering. What about the converse? Find a logistic map for which some non-wandering point is not recurrent.

- For an irrational rotation ρ_α of the circle by the angle α, develop a symbolic description of the dynamics of ρ_α by partitioning the circle into the sets $A = [0, \alpha]$ and $B = [\alpha, 1]$ (expressed in units of turns). Let Σ_α be the space of symbolic sequences and σ_α be the shift map.

  1) Examine the structure of orbits under ρ_α and σ_α.
  2) Although ρ_α is not sensitive to initial conditions, σ_α does not sensitively depend on initial conditions. Show this.
  3) Explain the structural difference between the two maps.

- Examine and develop a piece of theory—for instance, basin boundaries as fractals (one or two dimensions), stochastic fractals, fractal dimension—or particularly interesting exercise from the text, especially one of the challenges at the end of a chapter.

Two-dimensional dynamics

- Say you have two equations
$$F(x, y) = 0 \qquad G(x, y) = 0$$
  in two *real* variables x, y. Using calculus of 2-variables, describe Newton's method for finding the common solutions to these equations. Explore the local and global dynamics. Compare this problem with Newton's method in one real variable. What makes it so much more difficult?

- The meteorological model of Lorenz is a fairly simple system of equations that describe circulation of air in the atmosphere. This circulation has the character of a leaky water-wheel (the rate of leakiness is a parameter that can change). Of course, these equations give rise to chaotic behavior. Investigate this model, paying particular attention to the development of chaos. Discuss what this says about weather prediction. Perhaps explain how the water-wheel model captures the atmospheric phenomena.
- A pendulum that can swing in two independent directions can exhibit chaotic motion. Construct, describe, and analyze dynamically this peculiar device.
- Take a solid torus and map it, in a one-to-one fashion, strictly inside of itself while winding around the torus's axis *twice*. Under iteration of this map, a complicated "curve-like" attractor appears—called a solenoid. Work out some of the basic properties and structure of the solenoid attractor.

  Reference: R. Devaney, *An Introduction to Chaotic Dynamical Systems*
- Study the basic questions and theory of *billiard dynamics*. Develop and explain some aspect of the subject—for instance, billiards on triangles. Conduct some suggestive and illustrative computational experiments.

  References:

  S. Tabachnikov, *Geometry and Billiards*

  M. Berger, *Geometry Revealed*, Ch. XI, "Geometry and dynamics I: Billiards"

  A. Baxter and R. Umble, "Periodic orbits of billiards on an equilateral triangle"
- One of the classical pieces of dynamical theory concerns planetary trajectories. Investigate the dynamics of objects in gravitational orbits. In particular, consider the stability of the arrangement when there are two bodies and when there are three.

  References:

  D. Acheson, *From Calculus to Chaos*

  J. Barrow-Green, *Poincaré and the Three Body Problem*
- Create a computer algorithm for (select one):
  - Making colored plots that show basins of attraction for (either one or two dimensional) maps or a family of maps
  - Plotting the stable and unstable manifolds of a two-dimensional saddle point (see Example 10.5, p. 407)
  - Computing and plotting of backward trajectories for non-invertible maps.
- Construct a *magnetic* pendulum—where some stationary arrangement of magnets repel (attract?) a swinging bob. Explore the dynamics of the system. In particular, can you find a model (that is, a map) whose iteration describes the physical process?