

Is the Solar System stable ?

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Résumé. Since the formulation of the problem by Newton, and during three centuries, astronomers and mathematicians have sought to demonstrate the stability of the Solar System. Thanks to the numerical experiments of the last two decades, we know now that the motion of the planets in the Solar System is chaotic, which prohibits any accurate prediction of their trajectories beyond a few tens of millions of years. The recent simulations even show that planetary collisions or ejections are possible on a period of less than 5 billion years, before the end of the life of the Sun.

1. Historical introduction ¹

Despite the fundamental results of Henri Poincaré about the non-integrability of the three-body problem in the late 19th century, the discovery of the non-regularity of the Solar System's motion is very recent. It indeed required the possibility of calculating the trajectories of the planets with a realistic model of the Solar System over very long periods of time, corresponding to the age of the Solar System. This was only made possible in the past few years. Until then, and this for three centuries, the efforts of astronomers and mathematicians were devoted to demonstrate the stability of the Solar System.

1.1. Solar System stability

The problem of the Solar System stability dates back to Newton's statement concerning the law of gravitation. If we consider a unique planet around the Sun, we retrieve the elliptic motion of Kepler, but as soon as several planets orbit around the Sun, they are subjected to their mutual attraction which disrupts their Kleperian motion. At the end of the volume of *Opticks* (1717,1730), Newton himself expresses his doubts on this stability which he believes can be compromised by the perturbations of other planets and also of the comets, as it was not known at the time that their masses were very small.

1. This part is adapted from a lecture of the author on October 19th 2006, at Lombardo Institute (Milan) in honor of Lagrange : *Lagrange et la stabilité du Système solaire* (Laskar, 2006).

And to show that I do not take Gravity for an essential Property of Bodies, I have added one Question concerning its Cause, chosing to propose it by way of a Question, because I am not yet satisfied about it for want of Experiments.

...

For while comets move in very excentrick orbs in all manner of positions, blind fate could never make all the planets move one and the same way in orbs concentrick, some inconsiderable irregularities excepted, which may have risen from the mutual actions of comets and planets upon one another, and which will be apt to increase, till this system wants a reformation.

These planetary perturbations are weak because the masses of the planets in the Solar System are much smaller than the mass of the Sun (Jupiter's mass is about 1/1000 of the mass of the Sun). Nevertheless, one may wonder as Newton whether their perturbations could accumulate over very long periods of time and destroy the system. Indeed, one of the fundamental scientific questions of the 18th century was to first determine if Newton's law does account in totality for the motion of celestial bodies, and then to know if the stability of the Solar System was granted in spite of the mutual perturbations of planets resulting from this gravitation law. This problem was even more important as observations actually showed that Jupiter was getting closer to the Sun while Saturn was receding from it. In a chapter devoted to the *secular terms*, De la Lande reports in the first edition of his "Abrégé d'Astronomie" (1774) the problems that arose from these observations².

Kepler écrivait en 1625 qu'ayant examiné les observations de Régiomontanus et de Waltherus, faites vers 1460 et 1500, il avait trouvé constamment les lieux de Jupiter & de Saturne plus ou moins avancés qu'ils ne devaient l'être selon les moyens mouvements déterminés par les anciennes observations de Ptolémée & celles de Tycho faites vers 1600.

Kepler wrote in 1625, after having considered the observations of Regiomontanus and Waltherus made in 1460 and 1500, that he found consistently that the locations of Jupiter & Saturn were more or less advanced as they should be when their mean motions was determined according to ancient observations of Ptolemy & those of Tycho made around 1600.

Following the work of Le Monnier (1746a, b) which, according to De La Lande³

a démontré le premier, d'une manière suivie et détaillée, après un travail immense sur les oppositions de Saturne (Mémoire de l'Académie 1746), que non seulement il y a dans cette planète des inégalités périodiques dépendantes de la situation par rapport à Jupiter, mais que dans les mêmes configurations qui reviennent après cinquante-neuf ans, l'erreur des Tables va toujours en croissant.

demonstrated for the first time, in a detailed manner, after great work on the oppositions of Saturn (Mémoire de l'Académie 1746), that not only there are some periodic inequalities in this planet that depends on its position relative to Jupiter, but in the same configurations returning after fifty-nine years, the error in the Tables is always growing.

2. Translating french language of the XVIIIth century is not an easy matter, and the translations in english are provided here only to give a rapid view of the original text. The reader is welcome to propose some better translations to the author.

3. De la Lande, Tables Astronomiques de M. Halley pour les planètes et les comètes, Paris, 1759

136 *Tables planétaires.*

Tables

Table II.
Mouvements moyens de Saturne pendant les Années Juliennes.

Années	Longitude de Υ .				Équar. secul.		Aphélie			Anomalie moyenne			Nœud			Argument de Latitude.						
	S.	D.	M.	S.	D.	M.	S.	D.	M.	S.	S.	D.	M.	S.	S.	D.	M.	S.				
1	0	12	13	21	0	0	0	0	1	20	0	12	12	1	0	0	0	18	0	12	13	3
2	0	24	26	43	0	0	0	0	2	40	0	24	24	3	0	0	0	36	0	24	26	7
3	1	6	40	4	0	0	0	0	4	0	1	6	36	4	0	0	0	54	1	6	39	10
B. 4	1	18	55	26	0	0	0	0	5	20	1	18	50	6	0	0	1	12	1	18	54	14
5	2	1	8	48	0	0	0	0	6	40	2	1	2	8	0	0	1	30	2	1	7	18
6	2	13	22	9	0	0	0	0	8	0	2	13	14	9	0	0	1	48	2	13	20	21
7	2	25	35	30	0	0	0	0	9	20	2	25	26	10	0	0	2	6	2	25	33	24
B. 8	3	7	50	53	0	0	0	0	10	40	3	7	40	13	0	0	2	24	3	7	48	29
9	3	20	4	14	0	0	0	0	12	0	3	19	52	14	0	0	2	42	3	20	1	32
10	4	2	17	35	0	0	0	0	13	20	4	2	4	15	0	0	3	0	4	2	14	35
11	4	14	30	57	0	0	0	0	14	40	4	14	16	17	0	0	3	18	4	14	27	39
B. 12	4	26	46	19	0	0	0	0	16	0	4	26	30	19	0	0	3	36	4	26	42	43
13	5	8	59	41	0	0	0	0	17	20	5	8	42	21	0	0	3	54	5	8	55	47
14	5	21	13	2	0	0	0	0	18	40	5	20	54	22	0	0	4	12	5	21	8	50
15	6	3	26	23	0	0	0	0	20	0	6	3	6	23	0	0	4	30	6	3	21	53
B. 16	6	15	41	46	0	0	0	0	21	20	6	15	20	26	0	0	4	48	6	15	36	58
17	6	27	55	7	0	0	0	0	22	40	6	27	32	27	0	0	5	6	6	27	50	1
18	7	10	8	28	0	0	0	0	24	0	7	9	44	28	0	0	5	24	7	10	2	4
19	7	22	21	50	0	0	1	0	25	20	7	21	56	30	0	0	5	42	7	22	16	8
B. 20	8	4	37	12	0	0	1	0	26	40	8	4	10	32	0	0	6	0	8	4	31	12
40	4	9	14	24	0	0	2	0	53	20	4	8	21	4	0	0	12	0	4	9	2	24
60	0	15	51	56	0	0	5	0	1	20	0	12	31	36	0	0	18	0	0	13	33	36
80	8	18	28	48	0	0	9	0	1	46	40	8	16	42	8	0	24	0	8	18	4	48
100	4	23	6	0	0	1	4	0	2	13	20	4	20	52	40	0	30	0	4	22	36	0
200	9	16	12	0	0	5	6	0	4	26	40	9	11	45	20	0	1	0	9	15	12	0
300	2	9	18	0	0	12	5	0	6	40	0	2	2	38	0	1	30	0	2	7	48	0
400	7	2	34	0	0	22	2	0	8	53	20	6	23	30	40	0	2	0	7	0	24	0
500	11	25	30	0	0	34	8	0	11	6	40	11	14	23	20	0	2	30	11	23	0	0
600	4	18	36	0	0	50	0	0	13	20	0	4	5	16	0	3	0	0	4	15	36	0
70	9	11	42	0	1	8	1	0	15	33	20	8	26	8	40	0	3	30	9	8	12	0
800	2	4	48	0	1	29	0	0	17	46	40	1	17	1	20	0	4	0	2	0	48	0
900	6	27	54	0	1	52	6	0	20	0	0	6	7	54	0	4	30	0	6	23	24	0
1000	11	21	0	0	2	19	0	0	22	13	20	10	28	46	40	0	5	0	11	16	0	0
2000	11	12	0	0	9	16	1	0	1	14	26	40	9	27	33	20	0	10	0	11	2	0
3000	11	3	0	0	20	51	1	0	2	6	40	0	8	26	20	0	15	0	10	18	0	0
4000	10	24	0	0	37	4	4	0	2	28	53	20	7	25	6	40	0	20	0	10	4	0
5000	10	15	0	0	57	55	6	0	2	21	6	40	6	23	53	20	0	25	0	9	20	0
6000	10	6	0	0	83	24	4	0	4	13	20	0	5	22	40	0	1	0	0	9	6	0

FIGURE 1. Reproduction of the Tables of Halley *Recueil de Tables Astronomiques publié sous la direction de l'Académie Royale des Sciences et Belles-Lettres de Prusse, Vol. II, 1776.*

These observations led Halley to introduce a quadratic secular term in the mean longitudes of Jupiter and Saturn. The Tables of Halley became an authority during several decades, and were reproduced in various forms. In particular, by the Royal Academy of Prussia (1776) (Figs. 1, 2) during the period when Lagrange lived in Berlin. These apparent irregularities of Jupiter's and Saturn's motions constituted one of the most important scientific problems of the 18th century because it was a question of knowing if Newton's law do account for the motion of planets, and also of deciding on the stability of the Solar System. This led Paris Academy of Sciences to propose several prizes for the resolution of this problem. Euler was twice awarded a Prize of the Academy for these questions, in 1748 and 1752. In his last memoir (Euler, 1752), which laid the foundations of the methods of perturbations, Euler

Régitre des Tables.

V. Tables planétaires de *Halley* comparées avec les observations, par *Mr. Schulze*, & enrichies d'autres tables qui s'y rapportent.

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FIGURE 2. Table of content of *Recueil de Tables Astronomiques publié sous la direction de l'Académie Royale des Sciences et Belles-Lettres de Prusse, Vol. II, 1776*. It includes a paper from Lagrange where he summarizes the results of his memoir on the nodes and inclinations of the planets (1774-1778).

believed that he had demonstrated that Newton's law induces secular variations in the mean motion of Jupiter and Saturn, variations he found to be of the same sign, contrary to the observations. In reality, we know now that these results from Euler were wrong.

1.2. The 1766 memoir

It is still this important question that Lagrange is trying to solve in his 1766 memoir *Solution de différents problèmes de calcul intégral*, which appeared in Turin's memoirs :

TABLE DE LA VARIATION DES ÉLÉMENTS DE JUPITER ET DE SATURNE,
SUIVANT LA THÉORIE.

	JUPITER.	SATURNE.
Variation de la plus grande équation du centre.	+ 7",4254 n	- 32",6086 n
Variation de l'inclinaison à l'écliptique.	- 1",0030 n	+ 2",7449 n
Mouvement moyen de l'aphélie par rapport aux étoiles fixes.	+86",6311 n	+471",8632 n
Inégalité croissante dans le mouvement de l'aphélie.	+ 0",0262 n^2	+ 0",1141 n^2
Mouvement moyen des nœuds par rapport aux étoiles fixes.	+86",1075 n	-256",4655 n
Inégalité croissante dans le mouvement des nœuds.	+ 0",0513 n^2	- 0",0405 n^2
Inégalité croissante dans le mouvement en longitude.	+ 2",7402 n^2	- 14",2218 n^2

FIGURE 3. Results of Lagrange (1766) for the calculation of the secular inequalities. He found the quadratic term $2''.7402n^2$ in the mean longitude of Jupiter, and $-14''.2218n^2$ in the mean longitude of Saturn, where n is the number of revolutions of each planet.

je me bornerai à examiner ici, d'après les formules données ci-dessus, les inégalités des mouvements de Jupiter et Saturne qui font varier l'excentricité et la position de l'aphélie de ces deux planètes, aussi bien que l'inclinaison et le lieu du nœud de leur orbites, et qui produisent surtout une altération apparente dans leurs moyens mouvements, inégalités que les observations ont fait connaître depuis longtemps, mais que personne jusqu'ici n'a encore entrepris de déterminer avec toute l'exactitude qu'on peut exiger dans un sujet si important.

I will only consider here, according to the above formulas, the inequalities in the motions of Jupiter and Saturn which induce some variations in the eccentricity and the position of aphelion of these two planets, as well as the inclination and location of the node of their orbits, and mostly which induce some apparent alteration in their means motions, inequalities that the observations have been made long known, but that so far nobody has yet undertaken to determine with the accuracy that is required for such an important subject.

It is obvious that Lagrange does not believe in Euler's results. However, the care with which Lagrange conducted his own study of the same problem will not be sufficient. Although Lagrange's results are in agreement with the behavior of the observations (he actually found that Jupiter accelerates while Saturn is slowing down (fig. 3), his calculations are still incorrect. However, this memoir remains a milestone for the development of new methods of resolutions of differential equations (see the more detailed work of F. Brechenmacher 2007).

1.3. The invariance of the semi-major axis

It is finally Laplace, who will first demonstrate the secular invariance of the semi-major axes of planets, results he publishes in the *Memoires de l'Académie des*

Sciences de Paris in 1776. On the secular inequality of the semi-major axes, he writes :

Elle ne paraît pas cependant avoir été déterminée avec toute la précision qu'exige son importance. M. Euler, dans sa seconde pièce sur les irrégularités de Jupiter et de Saturne, la trouve égale pour l'une et l'autre de ces planètes. Suivant M. de Lagrange, au contraire, dans le troisième Volume des *Mémoires de Turin*, elle est fort différente pour ces deux corps. . . j'ai lieu de croire, cependant, que la formule n'est pas encore exacte. Celle à laquelle je parviens est fort différente. . . en substituant ces valeurs dans la formule de l'équation séculaire, je l'ai trouvée absolument nulle ; d'où je conclus que l'altération du mouvement moyen de Jupiter, si elle existe, n'est point due à l'action de Saturne.

It does not seem, however, to have been determined with all the precision required by its importance. Mr. Euler, in his second piece on the irregularities of Jupiter and Saturn, find it equal for both these planets. According to Mr. de Lagrange, on the contrary, the third volume of Mémoires de Turin, it is very different for these two bodies. . . I have some reasons to believe, however, that the formula is still not accurate. The one which I obtain is quite different. . . by substituting these values in the formula of the secular equation, I found absolutely zero, from which I conclude the alteration of the mean motion of Jupiter, if it exists, does not result from the action of Saturn.

Laplace's result is admirable, because he succeeds where the most outstanding intellects of the century, Euler and Lagrange, have failed, although they set up (with d'Alembert) the components which have permitted this discovery. Laplace's result is all the more striking as it runs counter to the observations, which, it is necessary to underline it, had not bothered Euler either. However Laplace does not call into question Newton's law of gravitation, but makes it necessary to find another cause for the irregularities of Jupiter and of Saturn. Luckily, there is another suitable culprit. Next to the planets, the movement of which seems regular and well ordered, other bodies exist, the comets, of which one had already noticed the very diverse trajectories. As their masses remained unknown at this time, one could evoke their attraction to explain any irregularity in the Solar System.

Il résulte de la théorie précédente que ces variations ne peuvent être attribuées à l'action mutuelle de ces deux planètes ; mais, si l'on considère le grand nombre de comètes qui se meuvent autour du Soleil, si l'on fait ensuite réflexion qu'il est très possible que quelques-unes d'entre elles aient passé assez près de Jupiter et de Saturne pour altérer leurs mouvements, . . il serait donc fort à désirer que le nombre des comètes, leurs masses et leurs mouvements fussent assez connus pour que l'on pût déterminer l'effet de leur action sur les planètes ; (Laplace, 1776a).

It follows from the above theory that these variations cannot be attributed to the mutual action of these two planets, but if we consider the large number of comets that move around the Sun, if we then imagine that it is very possible that some of them have passed close enough to Jupiter and Saturn to alter their motions, . . it would be very desirable that the number of comets, their masses and their movements were quite known so that we could determine the effect of their action on the planets (Laplace, 1776a).

The importance of the analysis of the comets' trajectories will also be fundamental for the interest that Laplace will take in the study of probability theory. He

indeed had to discriminate whether the variety of the trajectories of comets are the result of chance or not (Laplace, 1776b)

1.4. Inclinations and eccentricities

Laplace had presented his results concerning the invariance of semi-major axes to the Academy in 1773. The following year, in October 1774, Lagrange, then in Berlin, submitted to the Paris Academy of Sciences a new memoir about the secular motions of inclinations and nodes of the planets. In this memoir, appear for the first time the linear differential equations with constant coefficients that represent to the first order the averaged motion of the planetary orbits.

Ce Mémoire contient une nouvelle Théorie des mouvements des nœuds et des variations des inclinaisons des orbites des planètes, et l'application de cette Théorie à l'orbite de chacune des six planètes principales. On y trouvera des formules générales, par lesquelles on pourra déterminer dans un temps quelconque la position absolue de ces orbites, et connaître par conséquent les véritables lois des changements auxquels les plans de ces orbites sont sujets.

This Memoir contains a new theory for the motion of the nodes and variations of the inclinations of the orbits of the planets, and the application of this theory to the orbit of each of the six main planets. It contains general formulas by which one can determine, for any time value, the absolute position of any such orbits, and therefore know the true laws to which are subject the planes of these orbits.

One of the important elements in the resolution of these equations is the use of the Cartesian variables

$$s = \tan i \sin \Omega ; \quad u = \tan i \cos \Omega , \quad (1)$$

where i is the inclination, and Ω is the longitude of the node. These variables are almost the same as those which are still used today for the study of planetary motions. Lagrange provided here for the first time a quasi-periodic expression for the motion of the orbital plane of the planets which we can now write in a more synthetic way thanks to complex notation

$$u(t) + \sqrt{-1}s(t) = \sum_{k=1}^6 \beta_k \exp(\sqrt{-1}s_k t) . \quad (2)$$

The s_k are the eigenvalues of the matrix with constant coefficients of the linear secular System. Of course, Lagrange did not use the matrices formalism which will only be put in place much later (see Brechenmacher 2007), but he had to carry out the same computation of the eigenvalues of a 6 x 6 matrix in an equivalent way. In order to do this, he will proceed by iteration, beginning by the resolution of the Sun-Jupiter-Saturn system. It is impressive to see that in spite of the uncertainties concerning the values of the masses of the inner planets (Mercury, Venus and Mars)⁴, Lagrange obtained values of the fundamental frequencies of the secular system (s_k) that are very close to the present ones (Tab. 1).

4. Mercury and Venus do not possess satellites which could provide a good determination of the masses of the planet by applying the third law of Kepler. The satellites of Mars, Phobos and Deimos will only be discovered many years later, in 1877.

k	Lagrange (1774)	Laskar <i>et al.</i> , 2004
s_1	5.980	5.59
s_2	6.311	7.05
s_3	19.798	18.850
s_4	18.308	17.755
s_5	0	0
s_6	25.337	26.347

TABLE 1. Secular frequencies s_k for the motion of the nodes and inclinations of the planetary orbits. The values of Lagrange (1774) and modern values of (Laskar *et al.*, 2004) are given in arcseconds per year. It may be surprising that modern values give less significant digits than those of Lagrange, but the chaotic diffusion in the Solar System causes a significant change in these frequencies, making it vain for a precise determination of the latter. the secular zero frequency s_5 results from the invariance of the angular momentum.

At the Academy of Sciences in Paris, Laplace was very impressed by the results of Lagrange. He himself had temporarily left aside his own studies concerning the secular motion of the planetary orbits. He understood immediately the originality and the interest of Lagrange's work and submitted without delay a new memoir to the Academy, concerning the application of Lagrange's method to the motion of eccentricities and aphelions of planetary orbits (Laplace 1775).

Je m'étais proposé depuis longtemps de les intégrer ; mais le peu d'utilité de ce calcul pour les besoins de l'Astronomie, joint aux difficultés qu'il présentait, m'avait fait abandonner cette idée, et j'avoue que je ne l'aurais pas reprise, sans la lecture d'une excellent mémoire *Sur les inégalités séculaires du mouvement des nœuds et de l'inclinaison des orbites des planètes*, que M. de Lagrange vient d'envoyer à l'Académie, et qui paraîtra dans un des volumes suivants. (*Laplace, œuvres t VIII, p.355*)

I had proposed myself, since a long time to incorporate them, but the little utility of these calculations for the purposes of Astronomy, added to the difficulties that it presented, made me abandon this idea, and I confess that I would not have returned to it, without the lecture of the great paper "Sur les inégalités séculaires du mouvement des nœuds et de l'inclinaison des orbites des planètes" that M. Lagrange has just send to the Academy, and which will appear in one of the following volumes. (Laplace, œuvres t VIII, p.355)

What is surprising, is that Laplace's memoir, submitted in December of 1774, is very quickly published, in 1775, with the Academy's memoirs of 1772, while the original memoir of Lagrange will have to wait until 1778 to be published with the other memoirs of the year 1774. The application to eccentricities and to aphelion is in fact immediate, using the variables

$$l = e \cos \varpi ; \quad h = e \sin \varpi . \quad (3)$$

J'ai de plus cherché si l'on ne pourrait pas déterminer d'une manière analogue les inégalités séculaires de l'excentricité et du mouvement de l'aphélie, et j'y suis heureusement parvenu ; en sorte que je puis déterminer, non seulement les inégalités séculaires du mouvement des nœuds et de l'inclinaison des orbites des planètes, les seules que M. de Lagrange ait considérées, mais encore celles de l'excentricité et du mouvement des aphélies, et comme j'ai fait valoir que les inégalités du moyen mouvement et de la distance moyenne sont nulles, on aura ainsi une théorie complète et rigoureuse de toutes les inégalités séculaires des orbites des planètes. (*Laplace, œuvres t VIII, p.355*)

In addition, I have searched if one could determine similarly the secular inequalities of eccentricity and motion of the aphelion, and I happily succeeded, so that I can determine not only the secular inequalities of the motion of nodes and the inclination of the orbits of the planets, the only ones that were considered by Mr. Lagrange, but also those of the eccentricity and motion of aphelion, and as I have argued that the inequalities of the mean motion and the average distance is zero, we will thus have a complete and rigorous theory of all secular inequalities of the orbits of the planets. (Laplace, œuvres t VIII, p.355)

One may be amazed that Laplace's memoir was published before Lagrange's, and Laplace himself feels obliged to add a note upon this

J'aurai dû naturellement attendre que les recherches de M. de Lagrange fussent publiées avant que de donner les miennes ; mais, venant de faire paraître dans les *Savants étrangers*, année 1773, un Mémoire sur cette matière, j'ai cru pouvoir communiquer ici aux géomètres, en forme de supplément, ce qui lui manquait encore pour être complet, en rendant d'ailleurs au Mémoire de M. de Lagrange toute la justice qu'il mérite ; je m'y suis d'autant plus volontiers déterminé, que j'espère qu'ils me sauront gré de leur présenter d'avance l'esquisse de cet excellent Ouvrage. (*Laplace, œuvres t VIII, p.355*)

I should have naturally waited that the research of Mr. Lagrange were published before to give mines, but as I just published in "Savants étrangers", year 1773, a Memoire on this matter, I thought I could communicate here to the geometers, in the form of a supplement, which was still lacking for completeness, giving back to the memoire of Mr. Lagrange all the justice it deserves ; I was even more resolute to do this, as I hope they would be grateful to me to present them in advance the sketch of this great work. (Laplace, œuvres t VIII, p.355)

Laplace sends his memoir to Lagrange who sends him back a long letter from Berlin on April, 10 1775 :

Monsieur et très illustre Confrère, j'ai reçu vos Mémoires, et je vous suis obligé de m'avoir anticipé le plaisir de les lire. Je me hâte de vous en remercier, et de vous marquer la satisfaction que leur lecture m'a donnée. Ce qui m'a le plus intéressé, ce sont vos recherches sur les inégalités séculaires. Je m'étais proposé depuis longtemps de reprendre mon ancien travail sur la théorie de Jupiter et de Saturne, de le pousser plus loin et de l'appliquer aux autres planètes ; j'avais même dessein d'envoyer à l'Académie un deuxième Mémoire sur les inégalités séculaires du mouvement de l'aphélie et de l'excentricité des planètes, dans lequel cette matière serait traitée d'une manière analogue à celle dont j'ai déterminé les inégalités du mouvement du nœud et des inclinaisons, et j'en avais déjà préparé les matériaux ; mais, comme je vois que vous avez entrepris vous-même cette recherche, j'y renonce volontiers, et je vous sais même très bon gré de me dispenser de ce travail, persuadé que les sciences ne pourront qu'y gagner beaucoup.

Mr. and illustrious Colleague, I received your memoirs, and I am obliged to you to have anticipated the pleasure of reading it. I look forward to thank you, and to mark the satisfaction their reading has given me. What I was most interested in, are your research on the secular inequalities. I thought long ago to take back my old work on the theory of Jupiter and Saturn, to push it further and apply it to other planets. I even planned to send a second memory Academy on the inequalities of the secular motion of the aphelion and eccentricity of the planets, in which the material is treated in a similar manner as what I have determined for the motion of the inequalities of the node and inclinations, and I had already prepared the materials, but as I see that you have undertaken yourself this research, I happily renounce to it, and I even thanks you for dispensing me of this work, convinced that science will largely gain from this.

So Lagrange specifies that he also had understood that the problem of the eccentricities could be treated in the same way, and because Laplace now deals with this question, Lagrange proposes to give up this subject to him. In fact, this “promise” will not last, and he sends back to d’Alembert a letter dated from May 29, 1775 which shows that he cannot resist continuing his research on this fascinating subject.

Je suis près à donner une théorie complète des variations des éléments des planètes en vertu de leur action mutuelle. Ce que M. de la Place a fait sur cette matière m’a beaucoup plu, et je me flatte qu’il ne me saura pas mauvais gré de ne pas tenir l’espèce de promesse que j’avais faite de la lui abandonner entièrement ; je n’ai pas pu résister à l’envie de m’en occuper de nouveau, mais je ne suis pas moins charmé qu’il y travaille aussi de son côté ; je suis même fort empressé de lire ses recherches ultérieures sur ce sujet, mais je le prie de ne m’en rien communiquer en manuscrit et de ne me les envoyer qu’imprimées ; je vous prie de bien vouloir le lui dire, en lui faisant en même temps mille compliments de ma part.

I am ready to give a complete theory for the variations of the elements of the planets under their mutual action. That Mr. de la Place did on this subject I liked, and I flatter myself that he will not be offended if I do not hold the kind of promise that I made to completely abandon this subject to him ; I could not resist to the desire to look into it again, but I am no less charmed that he is also working on it on his side ; I am even very eager to read his subsequent research on this topic, but I do ask him not to send me any manuscript and send them to me only in printed form ; I would be obliged that you tell him, with a thousand compliments from my side.

Indeed, Lagrange resumed his work and published his results in several memoirs in 1781, 1782, 1783a, b, and 1784 in which he gives the first complete solution of the motion of the six main planets. Perhaps due to the deception he felt following the submission of his article of 1774 to Paris Academy of Sciences, he chose this time to publish his works in the Memoirs of the Academy of Berlin.

1.5. The great inequality of Jupiter and Saturn ⁵

Laplace had demonstrated the invariance of secular variations of the semi-major axes, considering only the first terms of the expansion of their average perturbations, but the problem of the accordance with observations remained. He resumed his search with his theory of Jupiter and Saturn as a base. A first element put him on

5. A more detailed account of this quest can be found in (Wilson, 1985), and (Laskar, 1992)

the right track : the observation of the energy's conservation in the Sun-Jupiter-Saturn system. If Newton's law is correct, the conservation of the system's energy implies that when one of the mean motion increases, the other must decrease. This is clearly observed. By neglecting the terms of order 2 compared to the masses, he found out that the quantity

$$\frac{m_J}{a_J} + \frac{m_S}{a_S} \quad (4)$$

must remain constant. With Kepler's law ($n^2 a^3 = Cte$) it gives :

$$\frac{dn_S}{dn_J} = -\frac{m_J}{m_S} \sqrt{\frac{a_J}{a_S}} \quad (5)$$

where for every planet Jupiter (J) or Saturn (S), m is the mass, a the semi-major axis and n the mean motion. Using the observations available at the time (Tab. 2), one finds $dn_S/dn_J = -2.32$, which is translated by Laplace as "Saturn deceleration must be compared to the acceleration of Jupiter, roughly, as 7 is with 3". Using the values obtained by Halley by comparison with the observations, one obtains $dn_S/dn_J = -2.42$, allowing Laplace to think with great confidence that "the variations observed in the motions of Jupiter and of Saturn result from their mutual action". Newton's law does thus not seem to be challenged, but it remained necessary to find the reason for these variations from Newton's equations. As Laplace demonstrated that there are no secular terms in the first order equations of semi major axes, he inferred that these changes of the average motion of the planets are probably due to short period terms (periodic terms with frequencies that are integer combinations of the mean motions of Jupiter and Saturn) which would be of period that is long enough to look like a secular term. A good candidate for this is the term associated to the combination of longitudes $2\lambda_J - 5\lambda_S$, with period of about 900 years.

planète	$1/m$	a (UA)	n ("/365j)
Jupiter	1067.195	5.20098	109182
Saturne	3358.40	9.54007	43966.5

TABLE 2. Values of the parameters of Jupiter and Saturn used in Laplace's work (Laplace, 1785).

This research led Laplace to undertake the construction of a more complete theory of the motion of the Jupiter-Saturn couple. After very long calculations, because to obtain these terms it is necessary to develop the perturbations to a high degree with respect to the eccentricities of Jupiter and Saturn, he obtained the following formulas (reduced here to their dominant terms) for the mean longitudes of Jupiter and Saturn :

$$\begin{aligned} \lambda_J &= n_J t + \epsilon_J + 20' \quad \sin(5n_S t - 2n_S t + 49^\circ 8' 40'') \\ \lambda_S &= n_S t + \epsilon_S + 46' 50'' \sin(5n_S t - 2n_S t + 49^\circ 8' 40'') \end{aligned} \quad (6)$$

ϵ_J and ϵ_S being the initial conditions for 1700 after J.C. Laplace then corrected the values of the mean motions of Jupiter and Saturn n_J and n_S with respect to

Halley's tables. He was then able to compare his new theory, without secular terms in the mean motions (or what is equivalent, without quadratic terms in the mean longitude) to modern and ancient observations. The differences in longitude between his theory and those of new observations (from 1582 to 1786) were all less than $2'$, while the differences with Halley's tables reached more than $20'$. He also compared his theory with the Chaldean observations of Saturn in 228 BC and of Jupiter at 240 BC transmitted by Ptolemy in the *Almageste*. These observations are of particularly good quality, because they identify precisely the planet positions in comparison with known stars. Laplace found a difference with his formulas only $55''$ for the first and $5''$ for the second.

The new theory of the Jupiter-Saturn couple that Laplace completed was therefore in perfect agreement with the observations from 240 BC to 1715 AD, without the necessity of an empirical secular term in the mean motion. The whole theory was entirely derived from Newton's law of gravitation. Laplace saw here *a new proof of the admirable theory of universal gravity*. He also obtained an important side result, that is, the mass of comets is certainly very small, otherwise their perturbations would have disturbed Saturn's orbital motion.

After this work, the secular terms of the mean motions will once and for all disappear from the astronomical tables, and in the second edition of his *Abrégé d'Astronomie*, De la Lande (1795) will reduce the chapter about the secular equations to a simple paragraph, recalling that Laplace's calculations on the great inequality of Jupiter and Saturn *make the acceleration of the one (Jupiter) and the delaying of the other (Saturn) disappear; their effect is only to make the duration of their revolutions to seem more or less long during nine centuries*.

1.6. Back to the semi-major axis

Laplace's demonstration on the secular invariance of the semi-major axes of the planets considered the expansion up to degree two in eccentricity and inclinations of the perturbing potential of the planets. Lagrange went back to this problem in 1776 using his method of variations of constants, which allowed him to redo the demonstration, without expansion in eccentricity, and therefore valid for all eccentricities. His demonstration is also particularly simple and very close to the current demonstration. Lagrange will once again come back over this problem in 1808 after Poisson had presented his famous Memoir of about 80 pages (Poisson 1808), where he showed that the invariance of the semi-major axes of the planets is still valid at the second order with respect to the masses.

Lagrange was in Paris during this time, Member of the Institute, where he had been called by Laplace, in 1787, as a *subsidized veteran of the Academy of Sciences*. In this memoir of 1808, Lagrange showed that using coordinates referring to the barycenter of the Solar System instead of using, as was previously the case, heliocentric coordinates, he succeeded in giving a more symmetric shape to the equations and considerably simplified Poisson's demonstration.

Indeed, he derived the differential equations of motion from a single function, and this was the beginning of Lagrangian formalism of variations of constants which already begun to express its considerable power in this difficult problem. This study conducted to the general method of Lagrange, described in the *Mémoire sur la théorie générale de la variation des constantes arbitraires dans tous les problèmes*

de la mécanique (Lagrange 1808 1809). This problem of the Solar System's stability and of the calculus of the secular terms observed by the astronomers was therefore fundamental in the development of mechanics and perturbative methods and more generally in the development of science in the XVIIIth century.

1.7. The *proof of stability of Lagrange and Laplace and Le Verrier's question*

After the work of Lagrange and Laplace, the stability of the Solar System seemed to be acquired. The semi-major axes of the orbits had no long-term variations, and their eccentricities and inclinations showed only small variations which did not allow the orbits to intersect and planets to collide. However, it should be noted that Lagrange and Laplace solutions are very different from Kepler's ellipses : the planetary orbits are no longer fixed. They are subject to a double movement of precession with periods ranging from 45000 years to a few million years : precession of the perihelion, which is the slow rotation of the orbit in its plan, and precession of nodes, which is the rotation of the orbital plane in space.

Lagrange and Laplace have written the first *proof* of the stability of the Solar System. But as Poincaré (1897) emphasizes in a general audience paper about the stability of the Solar System :

Les personnes qui s'intéressent aux progrès de la Mécanique céleste, ... , doivent éprouver quelque étonnement en voyant combien de fois on a démontré la stabilité du système solaire.

Lagrange l'a établie d'abord, Poisson l'a démontrée de nouveau, d'autres démonstrations sont venues depuis, d'autres viendront encore. Les démonstrations anciennes étaient-elles insuffisantes, ou sont-ce les nouvelles qui sont superflues ?

L'étonnement de ces personnes redoublerait sans doute, si on leur disait qu'un jour peut-être un mathématicien fera voir, par un raisonnement rigoureux, que le système planétaire est instable.

Those who are interested in the progress of celestial mechanics, ... must feel some astonishment at seeing how many times the stability of the Solar System has been demonstrated.

Lagrange established it first , Poisson has demonstrated it again, other demonstrations came afterwards, others will come again. Were the old demonstrations insufficient, or are the new ones unnecessary ?

The astonishment of those people would probably double, if they would be told that perhaps one day a mathematician will demonstrate, by a rigorous reasoning, that the planetary system is unstable.

In fact, the work of Lagrange and Laplace concerned only the linear approximation of the average motion of the planets. In modern language, we can say they demonstrated that the origin (equivalent to planar circular motions) is an elliptical fixed point in the secular phase space, obtained after averaging of order one with respect to the mean longitudes. Later on, Le Verrier (1840, 1841) resumed the computations of Lagrange and Laplace. This was before his discovery of Neptune in 1846, from the analysis of the irregularities in the motion of Uranus. In a first paper (1840) he computed the secular system for the planets, following the previous works of Lagrange and Laplace, with the addition of the computation of the change in the solutions resulting from possible new determinations of the planetary masses. Soon

after (1841), he reviewed the effects of higher order terms in the perturbation series. He demonstrated that these terms produce significant corrections to the linear equations, and that the calculations of Laplace and Lagrange could not be used for an indefinite period. Le Verrier (1840, 1841) raised the question of the existence of small divisors in the secular system of the inner planets. This was even more important as some of the values of the planetary masses were very imprecise, and a change of the mass values could lead to a very small divisor, eventually equal to zero. The problem for Le Verrier was then that the terms of third order could be larger than the terms of second order, which in his view compromised the convergence of the solutions.

Ces termes acquièrent par l'intégration de très petits diviseurs ; et ainsi il en résulte, dans les intégrales, des termes dus à la seconde approximation, et dont les coefficients surpassent ceux de la première approximation. Si l'on pouvait répondre de la valeur absolue de ces petits diviseurs, la conclusion serait simple : la méthode des approximations successives devrait être rejetée

Through integration, these terms acquire very small divisors ; and thus it results in the integrals, some terms from the second approximation, and which coefficients exceed those of the first approximation. If we could bound the absolute value of these small divisors, the conclusion would be simple : the method of successive approximations should be rejected

The indeterminacy of the masses of the inner planets thus did not allow Le Verrier to decide of the stability of the system and he could only ask for the mathematicians' help to solve the problem.

Il paraît donc impossible, par la méthode des approximations successives, de prononcer si, en vertu des termes de la seconde approximation, le système composé de Mercure, Vénus, la Terre et Mars, jouira d'une stabilité indéfinie ; et l'on doit désirer que les géomètres, par l'intégration des équations différentielles, donnent les moyens de lever cette difficulté, qui peut très bien ne tenir qu'à la forme.

It seems thus impossible, by the method of successive approximations to decide whether, owing the terms of the second approximation, the system consisting of Mercury, Venus, Earth and Mars will enjoy indefinite stability and we must desire that geometers, by the integration of the differential equations of motion will provide the means to overcome this difficulty, which may well result only from the form.

1.8. Poincaré : the geometer's answer

But Poincaré (1892-99) will give a negative answer to Le Verrier's question. To do this, he completely re-thought the methods of celestial mechanics from the work of Hamilton and Jacobi. Poincaré demonstrated that it is not possible to integrate the equations of the movement of three celestial bodies subject to their mutual gravitational interaction, and that it is impossible to find an analytical solution representing the planetary motion, valid over an infinite time interval. In the same way, he concluded that the perturbations series used by astronomers to calculate the motion of the planets are not converging on an open set of initial conditions.

Poincaré therefore showed that the series of the astronomers are generally divergent. However, he had a high regard for the work of the astronomers of the time, and also pointed out that these divergent series can still be used as a very good

approximation for the motion of the planets for some time, which can be long, but not infinite. Poincaré did not seem to think that his results may have great practical importance, if not precisely for the study of the stability of the Solar System.

Les termes de ces séries, en effet, décroissent d'abord très rapidement et se mettent ensuite à croître ; mais, comme les astronomes s'arrêtent aux premiers termes de la série et bien avant que ces termes aient cessé de décroître, l'approximation est suffisante pour les besoins de la pratique. La divergence de ces développements n'aurait d'inconvénients que si l'on voulait s'en servir pour établir rigoureusement certains résultats, par exemple la stabilité du système solaire.

The terms of these series, in fact, decrease first very quickly and then begin to grow, but as the Astronomers's stop after the first terms of the series, and well before these terms have stop to decrease, the approximation is sufficient for the practical use. The divergence of these expansions would have some disadvantages only if one wanted to use them to rigorously establish some specific results, as the stability of the Solar System.

It should be noted that Poincaré means here stability on infinite time, which is very different from the practical stability of the Solar System, which only makes sense on a time comparable to its life expectancy time interval. Le Verrier had reformulated the question of the stability of the Solar System by pointing out the need to take into account the terms of higher degree than those considered by Laplace and Lagrange ; Poincaré is even more demanding, asking for the convergence of the series :

Ce résultat aurait été envisagé par Laplace et Lagrange comme établissant complètement la stabilité du système solaire. Nous sommes plus difficiles aujourd'hui parce que la convergence des développements n'est pas démontrée ; ce résultat n'en est pas moins important.

This result would have been considered by Laplace and Lagrange as establishing completely the stability of the Solar System. We are more demanding today because the convergence of expansions has not been demonstrated ; This result is nevertheless important.

Poincaré demonstrated the divergence of the series used by astronomers in their perturbations computations. As usual, he studied a much larger variety of perturbation series, but apparently not of immediate interest to astronomers, as they required to modify the initial conditions of the planets. However, Poincaré raise some doubts on the divergence of this type of series :

Les séries ne pourraient-elles pas, par exemple, converger quand x_1^0 et x_2^0 ont été choisis de telle sorte que le rapport n_1/n_2 soit incommensurable, et que son carré soit au contraire commensurable (ou quand le rapport n_1/n_2 est assujéti à une autre condition analogue à celle que je viens d'énoncer un peu au hasard) ? Les raisonnements de ce chapitre ne me permettent pas d'affirmer que ce fait ne se présentera pas. Tout ce qu'il m'est permis de dire, c'est qu'il est fort invraisemblable

The series could they not, for example, converge when x_1^0 and x_2^0 have been chosen so that the ratio n_1/n_2 is incommensurable, and its square is instead commensurable (or when the ratio n_1/n_2 is subject to another condition similar to that I have enounced somewhat randomly) The arguments of this chapter do not allow me to say that this does not exist. All I am allowed to say is that this is highly unlikely.

Half a century later, in line with the work of Poincaré, the Russian mathematician A. N. Kolmogorov actually demonstrated that these convergent perturbation series exists.

1.9. Back to stability

Kolmogorov (1954) analyzed again the problem of convergence of the perturbation series of celestial mechanics and demonstrated that for non-degenerated perturbed Hamiltonian systems, close to the non-regular solutions described by Poincaré, there are still regular quasiperiodic trajectories filling tori in the phase space. This result is not in contradiction with the result of non-integrability of Poincaré, because these tori, parameterized by the action variables, are isolated. This result has been completed by Arnold (1963a) which demonstrated that, for a sufficiently small perturbation, the set of invariant tori foliated by quasi-periodic trajectories is of strictly positive measure, measure that tends to unity when the perturbation tends to zero. Moser (1962) has established the same kind of results for less strong conditions that do not require the analyticity of the Hamiltonian. These theorems are generically called KAM theorems, and have been used in various fields. Unfortunately, they do not directly apply to the planetary problem that present some proper degenerescence (the unperturbed Hamiltonian depends only on the semi-major axis, and not on the other action variables (related to eccentricity and inclination)). This led Arnold to extend the proof of the existence of invariant tori, taking into account the phenomenon of degenerescence. He then applied his theorem explicitly to a planar planetary system with two planets, for a semi-major axes' ratio close to zero, then demonstrating the existence of quasiperiodic trajectories for sufficiently small values of the planetary masses and eccentricities (1963b). This result was later on extended to more general two planets spatial planetary systems (Robutel, 1995). More recently, Féjoz and Herman (2004) have shown the existence of tori of quasiperiodic orbits in a general system of N planets, but this result still requires extremely small planetary masses.

The results of Arnold brought many discussions, indeed, as the quasiperiodic KAM tori are isolated, an infinitely small variation of the initial conditions will turn an infinitely stable quasiperiodic solution into a chaotic, unstable solution. Furthermore, as the planetary system has more than two degrees of freedom, none of the KAM tori separates the phase space, leaving the possibility for the chaotic trajectories to travel great distances in the phase space. This is the diffusion phenomenon highlighted by Arnold.

In fact, later results showed that in the vicinity of a regular KAM torus, the diffusion of the trajectories is very slow (Nekhoroshev, 1977, Giorgilli al. 1989, Lochak, 1993, Morbidelli and Giorgilli, 1995), and may be negligible for a very long time, eventually as long as the age of the universe. Finally, although the masses of the actual planets are much too large for these results to be applied directly to the

Solar System⁶, it is generally assumed that the scope of these mathematical results goes much further than their demonstrated limits, and until recently it was generally accepted that the Solar System is stable, *to any reasonable acceptance of this term*. Over the last twenty years, the problem of the stability of the Solar System has considerably progressed, largely through the assistance provided by computers which allow extensive analytical calculations and numerical integrations of realistic models of the Solar System on durations that are now equivalent to his age. But this progress is also due as well to the understanding of the underlying dynamics, resulting from the development of the theory of dynamical systems since Poincaré.

2. Numerical computations

The orbital motion of the planets in the Solar System has a very privileged status. Indeed, it is one of the best modeled problems of physics, and its study may be practically reduced to the study of the behavior of the solutions of the gravitational equations (Newton's equations supplemented by relativistic corrections) by considering point masses, except in the case of the interactions of the Earth-Moon system. The dissipative effects are also very small, and even if we prefer to take into account the dissipation by tide effect in the Earth-Moon system to obtain a solution as precise as possible for the motion of the Earth (e.g. Laskar et al., 2004), we can very well ignore the loss of mass of the Sun. The mathematical complexity of this problem, despite its apparent simplicity (especially if it is limited to the Newtonian interactions between point masses) is daunting, and has been a challenge for mathematicians and astronomers since its formulation three centuries ago. Since the work of Poincaré, it is also well known that the perturbative methods that were used in the planetary calculations for almost two centuries cannot provide precise approximations of solutions on an infinite time. Furthermore, as indicated above, the rigorous results of stability by Arnold (1963ab) do not apply to realistic planetary systems.

Since the apparition of computers, the numerical integration of the planetary equations has emerged as a simple way to overcome this complexity of the solutions, but this approach has always been limited up to the present by computer technology. The first long numerical integrations of the Solar System orbits were limited to the outer planets, from Jupiter to Pluto (Cohen et al., 1973, Kinoshita and Nakai, 1984). Indeed, the more the orbital motion of the planets is fast, the more it is difficult to integrate them numerically, because the required integration step decreases with the period of the planet. Using a conventional numerical method, to integrate the orbit of Jupiter, a integration step size of 40 days is sufficient, whereas a 0.5 days is necessary to integrate the motion of the entire Solar System including Mercury. The first integrations of the outer planets system that were performed over 100 Myr and then 210 Myr (Carpino al, 1987, Nobili al, 1989, Applegate et al., 1986) essentially confirmed the stability of the outer planets system, finding quasiperiodic orbits similar to those of Lagrange or Le Verrier. It is only when Sussman and Wisdom (1988) have extended their calculations on 875 Myr that the first signs of instability in the motion of Pluto have appeared, with a Lyapunov time (the inverse

6. The application of Nekhoroshev theorem for the stability in finite time of the Solar System was made by Niederman (1996), but required planetary masses of the order of 10^{-13} solar mass.

of the Lyapunov exponent) of 20 Myr. But as the mass of Pluto (which is no longer considered as a *planet* since the resolution of the International Astronomical Union in 2006) is very low (1/130 000 000 the solar mass), this instability does not manifest itself by macroscopic instabilities in the remaining part of the Solar System, which appeared very stable in all these studies.

2.1. Chaos in the Solar System

Numerical integrations allow to obtain very accurate solutions for the trajectories of planets, but are limited by the short time step, necessary to achieve this precision in the case of the complete Solar System, where it is necessary to take into account the motion of Mercury, and even of the Moon. It should be noted that, until 1991, the only available numerical integration for a realistic model of the whole Solar System was the numerical integration of the Jet Propulsion Laboratory DE102 (Newhall et al., 1983), calculated over only 44 centuries.

I opted then for a different approach, using analytical perturbation methods, in the spirit of the work of Lagrange, Laplace, and Le Verrier. Indeed, since these pioneering works, the *Bureau des Longitudes*⁷, always has been the place of development of analytical planetary theories based on the classical perturbation series (Brumberg and Chapront, 1973, Bretagnon, 1974, Duriez, 1979). Implicitly, these studies assumed that the movement of celestial bodies is quasiperiodic and regular. These methods were essentially the same as those which were used by Le Verrier, with the additional help of computers for symbolic calculations. Indeed, these methods can provide very good approximations of the solutions of the planets over thousands of years, but they will not be able to provide answers to the questions of the stability of the Solar System. This difficulty, which has been known since Poincaré is one of the reasons that motivated the direct numerical integration of the equations of motion.

Nevertheless, the results of the KAM theorems suggested the possibility that classic perturbative solutions could be developed using computer algebra, to find quasiperiodic solutions of the orbital motions in the Solar System. However, seeking to build such a solution, I realized that the existence of multiple resonances in the averaged system of the inner planets rendered illusory such an approach (Laskar, 1984). This difficulty led me to proceed in two distinct stages :

The first step is the construction of an average system, similar to the systems studied by Lagrange and Laplace. The equations then do not represent the motion of the planets, but the slow deformation of their orbit. This system of equations, obtained by an averaging of order two over the fast angles (the mean longitudes) thanks to dedicated computer algebra programs, comprises 153824 polynomial terms. Nevertheless, it can be considered as a simplified system of equations, because its main frequencies are now the frequencies of precession of the orbit of the planets, and not their orbital periods. The complete system can therefore be numerically integrated with a very big step size of about 500 years. The averaged contributions of the Moon and of general relativity are added without difficulty and represent just a few additional terms (Laskar, 1985, 1986).

7. The *Bureau des Longitudes* was founded on 7 messidor an III (june, 25 1795) to develop astronomy and celestial mechanics. Its founding members were Laplace, Lagrange, Lalande, Delambre, Méchain, Cassini, Bougainville, Borda, Buache, Caroché.

The second step, namely the numerical integration of the average (or secular) system, is then very effective and could be performed over more than 200 Myr in only a few hours of computation time. The main result of this integration was to reveal that the whole Solar System, and specifically the inner Solar System (Mercury, Venus, Earth and Mars), is chaotic, with a Lyapunov time of about 5 million years (Laskar, 1989). An error of 15 meters in the initial position of the Earth gives rise to an error of about 150 meters after 10 Ma, but the same error becomes 150 million km after 100 Ma. It is therefore possible to construct precise ephemeris over a period of a few tens of Ma (Laskar et al., 2004, 2011), but it becomes practically impossible to predict the movement of the planets beyond 100 million years.

When these results were published, the only possible comparison was a comparison with the planetary ephemeris DE102, over only 44 centuries. This however allowed to be confident about the results, by comparing the derivatives of the averaged solutions at the origin (Laskar, 1986, 1990). At this time, there was no possibility of obtaining similar results by direct numerical integration.

Thanks to the rapid advances in computer industry, just two years later, Quinn et al. (1991) have been able to publish a numerical integration of the whole Solar System, taking into account the effects of general relativity and of the Moon, over 3 Myr in the past (later complemented by an integration from - 3 Myr to + 3 Myr). The comparison with the secular solution (Laskar, 1990) then showed a very good agreement, and confirmed the existence of secular resonances in the inner Solar System (Laskar *et al.*, 1992a). Later, using a symplectic integrator that allowed them to use a large step size for the numerical integration of 7.2 days, Sussman and Wisdom (1992) obtained an integration of the Solar System over 100 Myr, which confirmed the value of the Lyapunov time of approximately 5 Myr for the Solar System.

2.2. Planetary motions over several million years (Myr)

The variations of eccentricities and inclinations of the planetary orbits are clearly visible on a few Myr (Fig. 4). Over one million years, the solutions resulting from the perturbation methods of Lagrange and Le Verrier would already give a good estimate of these variations that are essentially due to the linear coupling in the secular equations. On several hundreds of Myr, the behavior of solutions of the external planets (Jupiter, Saturn, Uranus and Neptune) are very similar to the one of the first Myr, and the motion of these planets appears to be very regular, which has also been shown very accurately by means of frequency analysis (Laskar, 1990).

2.3. Planetary motions over several billion years (Gyr)

Once it is known that the motion of the Solar System is chaotic, with exponential divergence of trajectories that multiplies the error on the initial positions by 10 every 10 Myr, it becomes illusory to try to retrieve, or predict the movement of the planets beyond 100 Myr by the calculation of a single trajectory. However, one can make such a computation to explore the phase space of the system. The calculated trajectory should then only be considered as a possible trajectory among others after 100 Myr. In (Laskar, 1994), such calculations have even been pushed on several billion years to highlight the impact of the chaotic diffusion of the orbits. In figure 5 we no longer represent the eccentricities of the planets, but their maximum value, calculated on

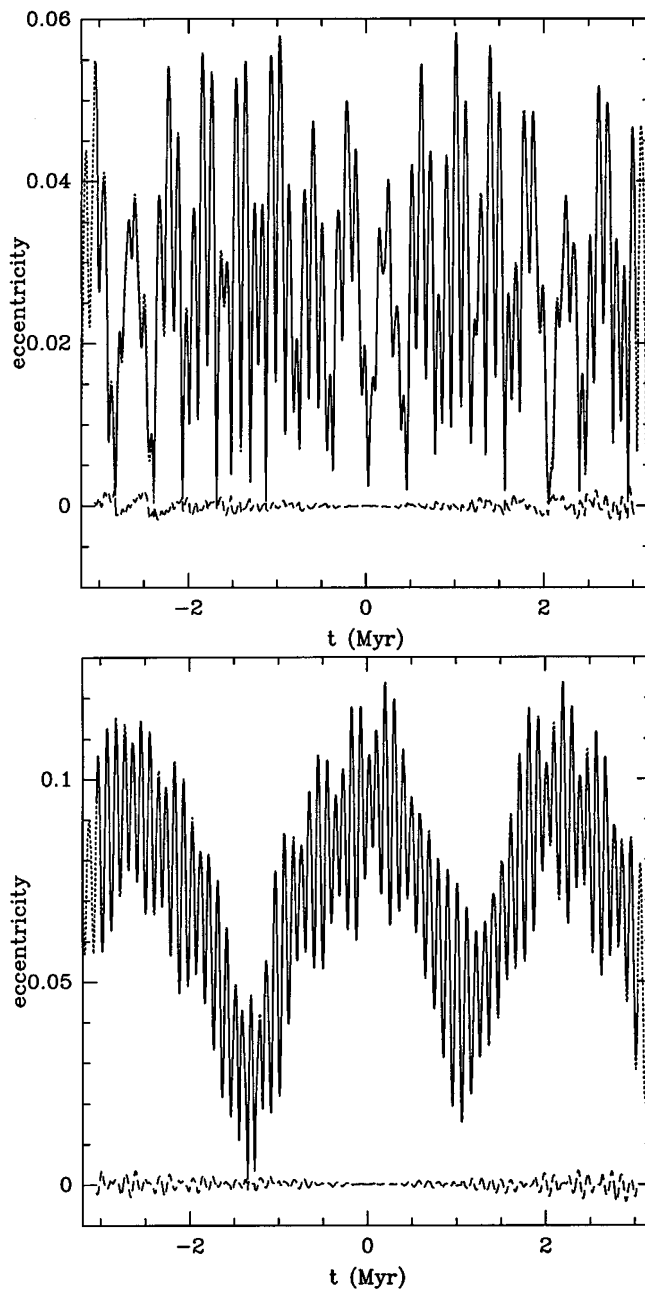


FIGURE 4. The eccentricity of the Earth (a) Mars and (b) from -3Myr to $+3\text{Myr}$. The solid line is the numerical solution from (Quinn *et al.*, 1991), and the dotted line, the secular solution (Laskar, 1990). For clarity, the difference between the two solutions is also plotted (Laskar *et al.*, 1992).

slices of 1 Myr. If the trajectory is quasiperiodic or close to quasiperiodic, this maximum will behave as a straight horizontal line, corresponding to the sum of the modulus of the amplitudes of the various periodic terms in the quasiperiodic expansion of the solution.

In this way, we are able to eliminate the oscillation of eccentricities resulting from the linear coupling already present in solutions of Lagrange or Le Verrier. The

remaining variations of this maximum, which appear in figure 6 are then the results of only the chaotic diffusion of the orbits. We see that for all the external planets (Jupiter, Saturn, Uranus, and Neptune), the maximum of the eccentricity is a horizontal line. It means that the motion of these planets is very close to quasiperiodic. However, for all the inner planets, there is a significant chaotic diffusion of the eccentricities. This diffusion is moderate for Venus and the Earth, important for Mars, whose orbit can reach eccentricities of the order of 0.2 (which does not allow collision with the Earth), and very strong for Mercury which reached an eccentricity of 0.5. This value is however not sufficient to allow for a collision with Venus, which requires an eccentricity of more than 0.7 for Mercury. But it is well understood that beyond 100 Myr, the trajectories of figure 6 only represent a possible trajectory of the Solar System, and a small change in initial conditions will significantly change the trajectories after 100 Myr.

To find out if collisions between Mercury and Venus are possible, it is therefore necessary to study the variations of the solutions under the influence of a small change in initial conditions. In (Laskar, 1994) I lead this study, using the secular system, showing that it was actually possible to build, section by section, an orbit of collision for Mercury and Venus. In a first step, the nominal trajectory is integrated over 500 Myr. Then, 4 additional trajectories are integrated, corresponding to small changes of 15 meters of the position reached at 500 Myr. All the trajectories are then integrated over 500 Myr and the trajectory of greater eccentricity of Mercury is retained and stopped in the neighborhood of the maximum of eccentricity. This operation is then repeated and leads to an eccentricity of Mercury of more than 0.9 in only 13 steps, in less than 3.5 Gyr, allowing thus a collisions with Venus. However, while repeating the same experiments with the trajectories of Venus, Earth and Mars, it was not possible to construct collisional solutions for these planets.

2.4. Chaotic diffusion in the Solar System

The 1994 approach however had some limitations, because the approximation obtained by the averaged equations decreases in accuracy as one approaches the collision. A study using the complete, non-averaged equations was therefore necessary to confirm these results. Despite the considerable increase in the power of computers since 1994, no complete study of this problem was conducted before 2009. Actually, because of the chaotic nature of the solutions, the only possible approach is a statistical study of a large number of solutions, with very similar initial conditions. This shows the difficulty of the problem. Indeed, before 2009, no direct integration of a single trajectory of the Solar System had yet been published using a realistic model, including the effect of the Moon and general relativity. To approach this problem, I have firstly carried out such a statistical study, using the averaged equations (whose numerical integration is about 1000 times faster than for the full equations), for 1000 different solutions that were integrated over 5 Gyr. This study (Laskar, 2008) showed that the probability to reach very high eccentricities for Mercury (> 0.6) is on the order of 1%. In this same study, I could also show that over periods of time longer than 500 Myr, the distributions of the eccentricities and inclinations of the inner planets (Mercury, Venus, Earth, and Mars) followed Rice's probability densities, and behave like random walks with a very simple empirical distribution law. These results differed significantly from the results published in 2002 by Ito and Tanikawa,

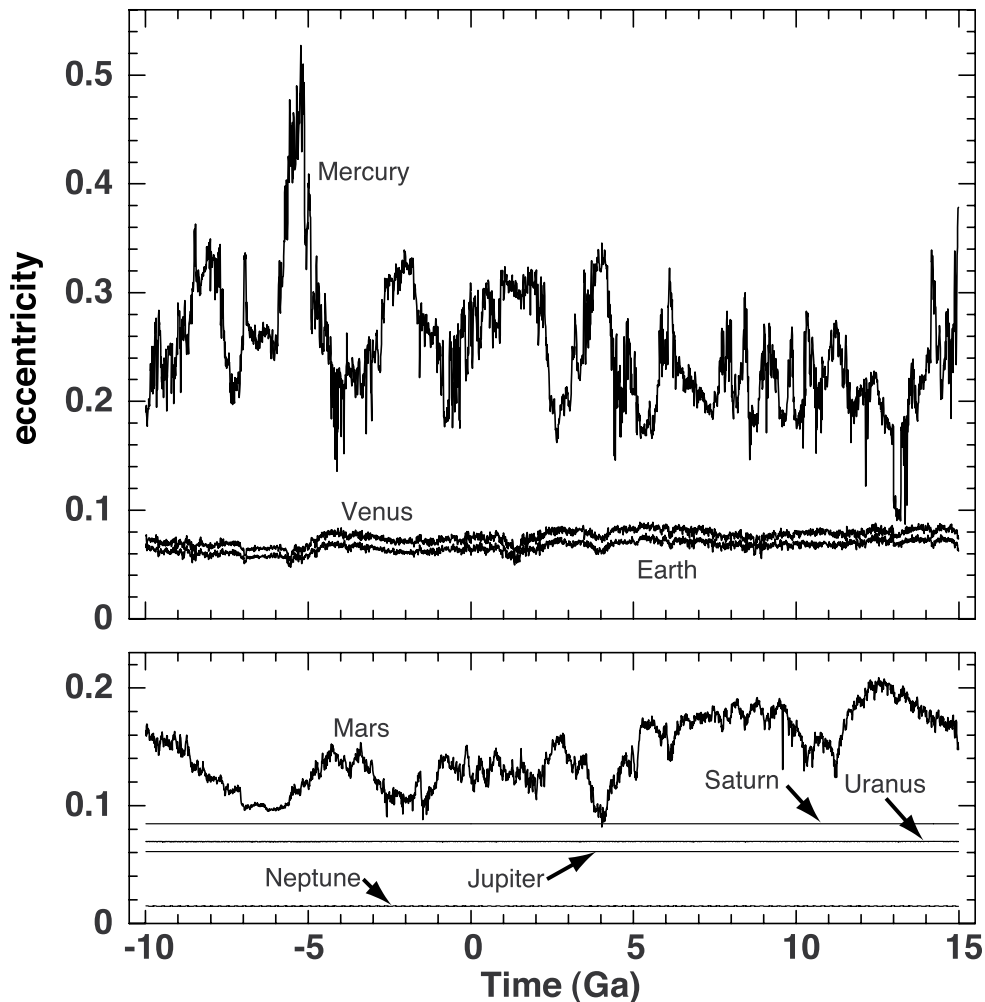


FIGURE 5. Numerical integration of the averaged equations of the Solar System from 10 to 15 Gyr. For each planet, only the maximum eccentricity reached on slices of 10 Myr is plotted. The motion of the large planets is very close to a quasiperiodic motion and the amplitude of the oscillations of their orbital elements does not vary. Instead, for all inner planets, there is a significant variation of the maximum eccentricity and inclination, which reflects the chaotic diffusion of the orbits (Laskar, 1994).

who had integrated 5 orbits on 5 Gyr for a purely (Non-relativistic) Newtonian model. I therefore also wanted to test the same statistics for a non-relativistic system, thinking that this system would be more stable, such as Ito's and Tanikawa's one (2002) who found for Mercury a maximum eccentricity of only 0.35. Much to my surprise, the result, on 1000 numerical solutions of the secular system with a pure Newtonian model on 5 Gyr revealed the opposite, and this system appeared far more unstable, with more than half of trajectories raising the eccentricity of Mercury up to 0.9.

To confirm these results, I have then proceeded to a direct integration, using a symplectic integrator (Laskar et al., 2004), of a pure Newtonian planetary model, for 10 trajectories with close initial conditions. The result was consistent with the

results of the secular system since 4 trajectories out of 10 led to eccentricity values for Mercury larger than 0.9 (Laskar, 2008). This large excursion of the eccentricity of Mercury is explained by the presence of a resonance between the perihelion of Mercury and Jupiter, which is made easier in the absence of general relativity (GR). It is known that GR increases the precession speed of the perihelion of mercury by $0.43''/yr$. This moves it from $5.15''/yr$ to $5.58''/yr$, and thus send it further from the value of the perihelion speed of Jupiter ($4.25''/yr$). Independently, Batygin and Laughlin (2008) published similar results shortly after. The American team, which resumes the calculation of (Laskar, 1994) on a system of non-relativistic equations, also demonstrated the possibility of collisions between Mercury and Venus. These results were still incomplete. Indeed, as the relativistic system is much more stable than the non-relativistic system, it is much more difficult to exhibit an orbit of collision between Mercury and Venus in the realistic (relativistic) system than in the non-relativistic system taken into consideration in these two previous studies. The real challenge was therefore in the estimation of the probability of collision of Mercury and Venus for a realistic, relativistic, model. It is precisely this program that I had in mind since the writing of my 2008 paper, that allowed me to estimate the probability of success of finding a collisional orbit for a realistic model of the Solar System.

2.5. The search for Mercury-Venus collisional orbits

With M. Gastineau, we then began a massive computation of orbital solutions for the Solar System motion, under various aspects, the ambition being to confirm and extend the results obtained 15 years before with the averaged equations. For this we used a non-averaged model consistent with the short-term highly accurate INPOP planetary ephemeris that we had developed in the past years (Fienga et al., 2008). Through the previous studies of the secular system, I had estimated to 3 million hours the computing time being necessary for such a study, but at the time, no national computing center was allowing even a 10 times smaller allocation of computing time. We then used all the means which we could have access to : local cluster of workstations, computing center of Paris Observatory, and a parallel machine that had just been installed at IPGP, Paris. To search for additional CPU time, we also undertook the development of the first full-scale application in astronomy on the EGEE grid with 500 cores (Vuerli and al., 2009). These different runs, associated with multiple difficulties due to the variety of machines and operating systems, nevertheless have allowed us to recover more than 2 million hours of CPU, but at the same time, a better estimate of the necessary computing time had increased the required time to more than 5 million hours. Quite fortunately, the availability of computing resources has changed in France in 2008, with the installation of the JADE supercomputer at CINES, near Montpellier, with more than 12000 cores⁸. As we could benefit of the experimental period on this machine, we started the computations as soon as the machine was switch on, in early August 2008, using 2501 cores, with one trajectory being computed on each core. We could then finalize our computations in about 6 months.

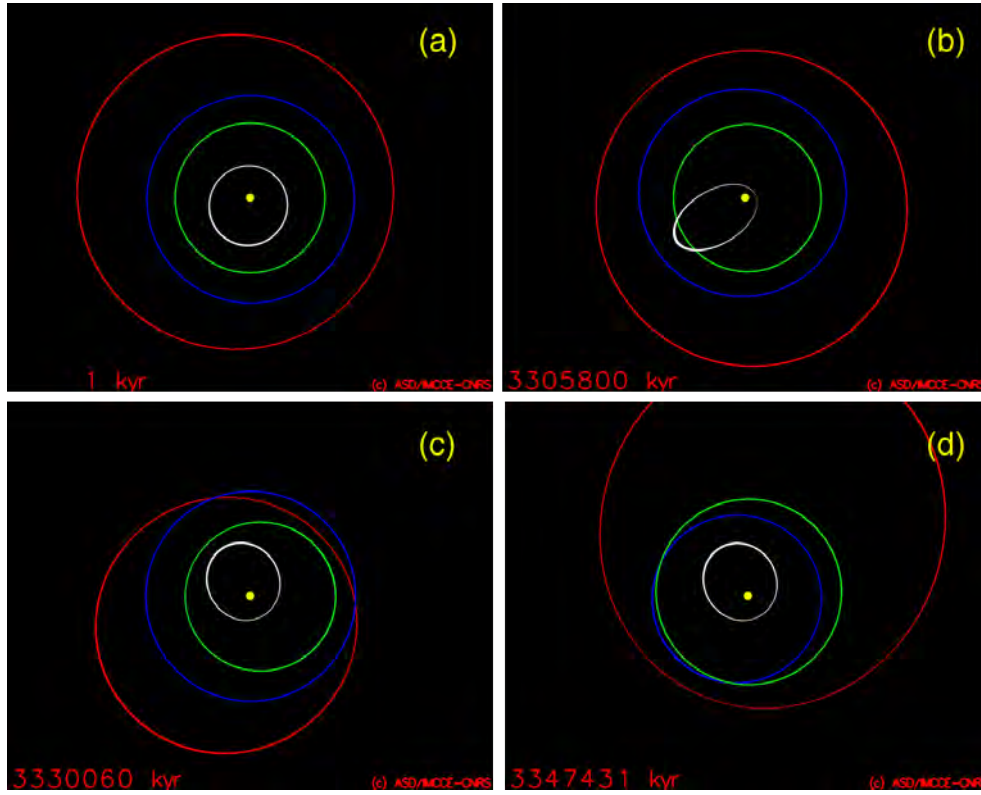


FIGURE 6. Example of long-term evolution of the orbits of the terrestrial planets: Mercury (white), Venus (green), Earth (blue), Mars (red). The time is indicated in thousands of years (kyr). (a) In the vicinity of the current state, the orbits are deformed under the influence of planetary perturbations, but without allowing close encounters or collisions. (b) In about 1 % of the cases, the orbit of Mercury can deform sufficiently to allow a collision with Venus or the Sun in less than 5 Gyr. (c) For one of the trajectories, the eccentricity of Mars increases sufficiently to allow a close encounter or collision with the Earth. (d) This leads to a destabilization of the terrestrial planets which also allow collisions between the Earth and Venus (Figure adapted from the results of the numerical simulations of Laskar and Gastineau, 2009).

2.6. Possibilité de collisions entre Mercure, Mars, Vénus et la Terre

With the JADE machine, we were able to simulate 2501 different solutions of the movement of the planets of the whole Solar System on 5 billion years, corresponding to the life expectancy of the system, before the Sun becomes a red giant. The 2501 computed solutions are all compatible with our current knowledge of the Solar System. They should thus be considered as equiprobable outcomes of the futur of the Solar System. In most of the solutions, the trajectories continue to evolve as in the current few millions of years : the planetary orbits are deformed and precess under the influence of the mutual perturbations of the planets but without the possibility of collisions or ejections of planets outside the Solar System. Nevertheless, in 1% of the cases, the eccentricity of Mercury increases considerably. In many cases, this deformation of the orbit of Mercury then leads to a collision with Venus, or with the

8. When this machine was installed, it was ranked 14 th worldwide among supercomputer centers.

Sun in less than 5 Ga, while the orbit of the Earth remained little affected. However, for one of these orbits, the increase in the eccentricity of Mercury is followed by an increase in the eccentricity of Mars, and a complete internal destabilization of the inner Solar System (Mercury, Venus, Earth, Mars) in about 3.4 Gyr. Out of 201 additional cases studied in the vicinity of this destabilization at about 3.4 Gyr, 5 ended by an ejection of Mars out of the Solar System. Others lead to collisions between the planets, or between a planet and the Sun in less than 100 million years. One case resulted in a collision between Mercury and Earth, 29 cases in a collision between Mars and the Earth and 18 in a collision between Venus and the Earth (Laskar and Gastineau, 2009). Beyond this spectacular aspect, these results validate the methods of semi-analytical averaging developed for more than 20 years and which had allowed, 15 years ago, to show the possibility of collision between Mercury and Venus (Laskar, 1994).

These results also answer to the question raised more than 300 years ago by Newton, by showing that collisions among planets or ejections are actually possible within the life expectancy of the Sun, that is, in less than 5 Gyr. The main surprise that comes from the numerical simulations of the recent years is that the probability for this catastrophic events to occur is relatively high, of the order of 1%, and thus not just a mathematical curiosity with extremely low probability values. At the same time, 99% of the trajectories will behave in a similar way as in the recent past millions of years, which is coherent with our common understanding that the Solar System has not much evolved in the past 4 Gyr. What is more surprising is that if we consider a pure Newtonian world, the probability of collisions within 5 Gyr grows to 60 %, which can thus be considered as an additional indirect confirmation of general relativity.

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