Dynamics of Functions with an Eventual Negative Schwarzian Derivative

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Abstract

In the study of one dimensional dynamical systems one often assumes that the functions involved have a negative Schwarzian derivative. In this paper we consider a generalization of this condition. Specifically, we consider the interval functions of a real variable having some iterate with a negative Schwarzian derivative and show that many known results generalize to this larger class of functions. The introduction of this class was motivated by some maps arising in neuroscience.

1 Introduction

Singer was the first to observe that if a function has a negative Schwarzian derivative then this property is preserved under iteration and moreover that this property puts restrictions on the type and number of periodic orbits the function can have [Sin78]. These properties, as well as those results derived from them, essentially rely on the global structure functions with a negative Schwarzian derivative have. For a full list of such properties see [dMvS93].

Later it was found that functions with a negative Schwarzian derivative also possess local properties useful in establishing certain distortion bounds. For the most part these properties are concerned with the way in which functions with a negative Schwarzian derivative increase cross ratios. Because of these special properties some results are known only in the case where a function has a negative Schwarzian derivative. However, an exception to this is the result by Kozlovski [Koz00] where it was shown that the assumption of a negative Schwarzian derivative is superfluous in the case of any C^3 unimodal map with nonflat critical point. For such functions there is always some interval around their critical point on which the first return map has a negative Schwarzian derivative. That is, the local structure of all such C^3 maps behave as maps with a negative Schwarzian derivative. More recently using the same technique van Strien and Vargas have generalized this result to multimodal functions [vSV04]. Also Graczyk, Sands and Swiatek have shown that any unimodal map with only repelling periodic points is analytically conjugate to a map with negative Schwarzian derivative [GSS01].

In this paper we consider those C^2 functions on a finite interval of the real line having some iterate with a negative Schwarzian derivative. This class, which we call functions with an *eventual negative Schwarzian derivative*, is broader than the classes of one-dimensional maps previously considered [Koz00, vSV04, GSS01]. In section 5 of this paper we give some examples of such functions which do not belong to those classes of one-dimensional maps previously studied. [1]

Since this class of functions contains those functions with a negative Schwarzian derivative as a subset we do not attempt to prove stronger results then have already been proved. Instead the strategy is to gather results that hold for functions with a negative Schwarzian derivative and show that these results hold for our more general functions. With this in mind the large majority of the results we will present will simply be restatements of those already known with the slight modification that only some iterate of our function need have a negative Schwarzian derivative. We note that this by no means is meant to be an exhaustive list of such results as the purpose of this paper is only to give further evidence that the useful properties possessed by functions with negative Schwarzian derivative are not limited to this family of functions.

This paper is organized as follows: In the next section we formally introduce the class of functions we are considering and present our main results. Section 3 presents the proof of those results that are topological in nature. Specifically, we prove the analogue of Singer's theorem and mention some import concepts and Corollaries that will be useful in what follows. Section 4 is comprised of those proofs of results which are more measure theoretic. Specifically, we generalize the main results in [BrLuSt03], [NowStr01], and [YoWa06] to this larger class of functions. In section 5 we give a partial characterization of functions that have an eventual negative Schwarzian derivative. That is, we give sufficient conditions under which a function will have an iterate with a negative Schwarzian derivative as well as some examples. The next section is devoted to an application of these results to a one-parameter family of maps that model the electrical activity in a neuronal cell near the transition to bursting [Med04, Med06]. Section 7 contains some concluding remarks.

2 Iterates and the Schwarzian Derivative

We consider only C^2 maps $f : I \to I$ on some compact interval I of real numbers and use the notation f' to denote the derivative with respect to spacial variable.

Definition 2.1 *A C^2 function $f : I \to I$ is said to have a negative Schwarzian derivative if on any open set of I, not containing critical points, $|f'|^{-1/2}$ is strictly convex.*

Definition 2.2 *We say a C^2 function $f : I \to I$ is eventually negatively Schwarzian with an eventual negative Schwarzian derivative if there exists $k \in \mathbb{N}$ such that f^k has a negative Schwarzian derivative. The smallest such number k is said to be the order of the derivative.*

It follows that functions with negative Schwarzian derivatives are themselves eventually Schwarzian.

[1] This class of eventually negatively Schwarzian one-dimensional maps was introduced by L. Bunimovich in an attempt to describe some one-dimensional maps which appear in the neurosciences [Med04, Med06]

Definition 2.3 *A C^2 map $f : I \to I$ is called S-multimodal if:*
(i) f has a finite number of critical points denoted by $\mathcal{C}(f)$.
(ii) For every $c \in \mathcal{C}(f)$, c is nonflat. That is, there exists some $\ell \in (1, \infty)$ such that

$$\lim_{x \to c} \frac{|f'(x)|}{|x - c|^{\ell - 1}} > 0, \tag{1}$$

where ℓ is the order of the critical point.
(iii) f has a negative Schwarzian derivative on I. If f has a single critical point c and $f''(c) \neq 0$ we call the map S-unimodal.

Definition 2.4 *A function $f : I \to I$ is eventually S-multimodal of order k or S^k-multimodal if there exists a smallest $k \in \mathbb{N}$ such that $f^k(x)$ is S-multimodal. We define S^k-unimodal similarly.*

We now state the main results of this paper the first of which is a generalization of Singer's Theorem to functions with an eventual negative Schwarzian derivative. That is, when we refer to Singer's Theorem we mean the theorem below where $k = 1$. For a proof see [Ced].

Theorem 2.5 *If $f : I \to I$ is an S^k-multimodal or unimodal map, then the immediate basin of attraction of any attracting periodic point contains either a critical point of f or a boundary point of I. Furthermore, any neutral periodic point of f is attracting, and there exist no interval of periodic points.*

One way to interpret this is to say that information about the number and orbits of critical points translates into information about periodic orbits. This first result is one of the things that sets this collection of functions apart from those considered elsewhere. In fact this property cannot be generalized by looking at first return maps in the sense that it is global in nature and as first return maps generically introduce discontinuities, i.e. more boundary points, this global structure is not preserved.

The next result strictly speaking is not a generalization of a known result. It is more accurately a Corollary to the main result in [BrLuSt03] using the restriction on the periodic orbits obtained in the previous theorem to simplify the hypothesis.

Denote $D_n(c) = |(f^n)'(f(c))|$.

Theorem 2.6 *Let $f : I \to I$ be C^3 with an eventual negative Schwarzian derivative, and a finite critical set $\mathcal{C}(f)$. If every point of $\mathcal{C}(f)$ has order $\ell \in (1, \infty)$ and f satisfies*

$$\sum_n D_n^{-1/(2\ell-1)}(c) < \infty \text{ for all } c \in \mathcal{C}(f) \tag{2}$$

then there exists an f-invariant probability measure μ absolutely continuous with respect to Lebesgue measure or an acip. Furthermore, some iterate of f is mixing and the correlation function C_n decays at the following rates:

Polynomial: If there is $C > 0$, $\alpha > 2\ell - 1$ such that $D_n(c) \geq Cn^{\alpha}$ for all $c \in \mathcal{C}(f)$ and $n \geq 1$, then for any $\tilde{\alpha} < \frac{\alpha-1}{\ell-1} - 1$ we have $C_n = \mathcal{O}(n^{-\alpha})$.

Exponential: If there is $C > 0$, $\beta > 0$ such that $D_n(c) \geq Ce^{\beta n}$ for all $c \in \mathcal{C}(f)$ and $n \geq 1$, then there is a $\tilde{\beta} > 0$ such that $C_n = \mathcal{O}(e^{-\tilde{\beta}})$.

In the original result by [BrLuSt03] the function was assumed to have no attracting or neutral periodic points. In effect the previous theorem says that instead of requiring that the function has no such periodic orbits we may assume that it has an eventual negative Schwarzian derivative.

However, if all that is needed is the existence of an acip or we have different orders of critical points then we may use the following result which generalizes the main result in [NowStr01] to functions with an eventual negative Schwarzian derivative.

Theorem 2.7 *If f is an S^k-multimodal or unimodal function and satisfies the condition*

$$\sum_n D_n^{-1/(\ell_{max})}(c) < \infty \text{ for all } c \in \mathcal{C}(f) \tag{3}$$

where ℓ_{max} is the largest order of the critical points in $\mathcal{C}(f)$ then f admits an absolutely continuous invariant probability measure.

In the unimodal case with the single critical point of order ℓ (3) is equivalent to

$$\sum_n D_n^{-1/(\ell)}(c) < \infty$$

which is less restrictive then condition (2) in Theorem 2.6.

The next theorem deals with one-parameter families of maps. In the theory presented in [YoWa06] there is a special class of functions denoted by $\mathcal{M}$ having strongly expansive properties. This set of functions is helpful in proving under what conditions one-parameter families of functions have absolutely continuous invariant measures for positive Lebesgue measure sets of parameters. Specifically, a technical but generically satisfied transversality condition in the parameter which will be denoted by (PT) is required for this to be the case. We refer the reader to the article for details.

Theorem 2.8 *Let $f_a : I \to I$ be a one-parameter family of C^3 functions where a belongs to some interval B of the real line. If, for some parameter value $\beta \in B$, f_β has a finite critical set $\mathcal{C}(f_\beta)$ and*
(i) f_β has an eventual negative Schwarzian derivative of order k
(ii) $f''_\beta(c) \neq 0$, $c \in \mathcal{C}(f_\beta)$
(iii) if $f^m_\beta(x) = x$, then $|(f^m_\beta)'(x)| > 1$
(iv) $inf_{i>0} d(f^i_\beta(c), \mathcal{C}(f_\beta)) > 0$, $c \in \mathcal{C}(f_\beta)$

then $f_\beta \in \mathcal{M}$ and has an absolutely continuous invariant measure. In particular if f_β^k satisfies the condition (PT) then on a positive Lebesgue measure set of parameters the family of functions f_a has an absolutely continuous invariant probability measure.

3 Topological Properties

In this section we prove Theorem 2.5 along with some corollaries that will be needed in the following sections but are of interest in their own right. We now give a proof of Theorem 2.5.

Proof: Let $f : I \to I$ be an S^k-multimodal or unimodal function of order $k > 1$. It follows from Singer's original Theorem that the immediate basin of attraction of any attracting periodic point of f^k contains either a critical point of f^k or a boundary point of I. To use this to our advantage suppose that $\tilde{x} \in I$ is an attracting periodic point of f^k. Then in its immediate basin of attraction B there is either an endpoint of I or a critical point $\tilde{c}$ of f^k. Since any point that is attracted to $\tilde{x}$ under f^k is also attracted to $\tilde{x}$ under f it follows that the immediate basin of attraction of $\tilde{x}$ under f contains B hence either an endpoint of I or $\tilde{c}$.

As $(f^k)'(\tilde{x}) = \prod_{i=0}^{k-1} f'(f^i(\tilde{x})) = 0$ if and only if $f^i(\tilde{x}) = 0$ for some $0 \leq i \leq k-1$ then the critical set of f^k is given by

$$\mathcal{C}(f^k) = \{x \in I | \ \exists\, c \in \mathcal{C}(f),\ 0 \leq i \leq k-1 \text{ where } f^i(x) = c\}.$$

That is $\mathcal{C}(f^k)$ is contained in the set of preimages of $\mathcal{C}(f)$ which we note here is finite since both $\mathcal{C}(f)$ and k are assumed finite. Then since $\tilde{c}$ is the preimage of some critical point of the function f the first statement of Theorem 2.5 follows from the fact that f has an attracting periodic orbit containing $\tilde{x}$ if and only if the same is true of f^k.

To verify the last two properties given in the theorem we note that if any neutral periodic orbit of f^k is attracting or if f^k has no intervals of periodic points then the same holds for f. Hence, the result again follows by a direct application of Singer's Theorem. □

Definition 3.1 *Let $f : I \to I$ be a C^1 multimodal map where C is the smallest subinterval of I that contains all critical points of f. If $f : I \to I$ is a C^1 unimodal map with critical point c let C be the interval with endpoints $f(c), f^2(c)$. We call the set*

$$J = \overline{\bigcap_{n \geq 0} \bigcup_{m \geq n} f^m(C)} \subseteq I$$

the dynamical interval of f.

It follows from the definition above that a point $x \in J$ if and only if there is an increasing sequence of the natural numbers $\{n_i\}$ such that $x \in f^{n_i}(C)$

for all $i \geq 1$ or that x is the limit point of such points. This implies that J consists of those points to which points of C return to infinitely many times under iteration. With a little thought this observation can be used to show that $f(J) = J$.

It should be pointed out that the term dynamical interval is somewhat misleading. It may be the case that the dynamic interval of a function is not an interval in the strict sense of being all the points between and including two numbers. For this and for other reasons we will consider a separate interval that contains the dynamical interval of a given function as a subset.

Definition 3.2 *Let $f : I \to I$ be a C^1 multimodal or unimodal map with critical set $\mathcal{C}(f)$. Let*

$$a = \inf_{n \geq 0}\{f^n(c) \mid c \in \mathcal{C}(f)\},\ b = \sup_{n \geq 0}\{f^n(c) \mid c \in \mathcal{C}(f)\}.$$

We call the interval $[a, b]$ the critical interval of the function f. That is, the smallest interval that contains the forward orbit of all critical points of f.

As we are concerned with the dynamics of the critical set $\mathcal{C}(f)$ we will often consider the function f restricted to its critical interval (see Lemma 3.4). This will in turn allow us to place certain restrictions on the dynamics of the function restricted to its dynamical interval.

Lemma 3.3 *Let $f : I \to I$ be a C^1 multimodal or unimodal map. Then the endpoints of the critical interval of f are either attracting or neutral periodic points of period 1 or 2 that attract some critical point of f or lie on the orbit of a critical point of f.*

Proof: The result follows from a straightforward consideration of the possible orbits of the critical points of f. For simplicity let $I = [0, 1]$ which can always be achieved by some affine change of coordinates. We first consider multimodal functions and let $I = L \cup C \cup R$ where $C = [c_l, c_r]$ is the smallest interval containing $\mathcal{C}(f)$, $L = [0, c_l]$, and $R = [c_r, 1]$. Also c_{max} and c_{min} are the critical point of f with largest and smallest function values respectively.

For the critical interval $[a, b]$ suppose in the following cases that b is not on the orbit of any critical point. Also note that on the intervals L and R that f is strictly monotonic.

Case 1: Let f be increasing on L and R. If $f^2(c_{max}) \leq f(c_{max})$ then it follows that $f([0, f(c_{max})]) \subseteq [0, f(c_{max})]$ so either $b = f(c_{max})$ or $b = c_r$. However, this violates the supposition that b is not on the orbit of any critical point. As it follows then that $f^2(c_{max}) > f(c_{max})$ then $c_r < f(c_{max})$ implying that f is strictly increasing on $[f^2(c_{max}), 1]$. Hence, by monotonicity any orbit containing a point in $[f^2(c_{max}, 1)]$ is attracted to a fixed point of this interval. The least of these fixed points p must be in b since c_{max} is attracted to it and as $f([0, p]) \subseteq [0, p]$.

Case 2: Let f be decreasing on L and R. Consider f^2 where we define L^2, C^2, R^2, c^2_{max} analogous to L, C, R and c_{max} for f^2. Note that $f^2(c^2_{max}) =$

$f(c_{max}) < b$. Also if f^2 is decreasing on L^2 this implies that $f(L) \subseteq [c_l, c_r]$ and f^2 decreasing on R^2 implies that $f(R) \subseteq [c_l, c_r]$. If either of these is the case then either $f(c_{max})$ or $f^2(c_{min})$ is equal to b. Since by the supposition neither of these is possible then f^2 is increasing on both L^2 and R^2 and the analysis reduces to that of Case 1. Therefore, c^2_{max} is attracted to b which implies the critical point c_{max} of f is attracted to a two cycle of f containing b.

Case 3: Let f be decreasing on L and increasing on R. If $f^2(c_{max}) \leq f(c_{max})$ and $f^2(c_{min}) \leq f(c_{max})$ then $f([f(c_{min}), f(c_{max})]) \subseteq [f(c_{min}), f(c_{max})]$ implying b lies on the orbit of c_{max} or c_r, which violates the supposition. If $f^2(c_{max}) > f(c_{max})$ then as in Case 1 either c_{min} or c_{max} is attracted to a fixed point in the interval $[f^2(c_{max}, 1)]$ which must be b. If $f^2(c_{min}) > f(c_{max})$ but $f^2(c_{max}) \leq f(c_{max})$ then $f^2(c_{min}) = b$ which is not allowed by the supposition

Case 4: Let f be increasing on L and decreasing on R. This however implies that $b = f(c_{max})$, or once again the supposition is violated.

Repeating this argument with appropriate modifications implies the same is true for the endpoint a. The result then follows for multimodal maps. For convex unimodal maps the situation is much simpler and depends only on whether the critical point c is greater than $f(c)$. If it is then the critical interval of f is given by $[f^2(c), f(c)]$ or $[c, f(c)]$. If $f(c) \leq c$ then the critical interval is given by $[a, c]$ where a is rightmost fixed point of f on I. In both cases the endpoints of the critical interval of f lie on the orbit of the critical point and the result follows since by an affine change of coordinates any unimodal function can be made convex. □

Lemma 3.4 *Let $f : I \to I$ be a C^1 multimodal or unimodal map. Then the dynamical interval of f is contained in its critical interval $\tilde{I}$. Moreover, $f(\tilde{I}) \subseteq \tilde{I}$.*

Proof: Let $[a, b]$ be the critical interval of f. Firstly, we claim that $f([a, b]) \subseteq [a, b]$. If this were not the case then suppose for some $\tilde{x} \in [a, b]$ that $f(\tilde{x}) > b$. Since for all $c \in \mathcal{C}(f)$ $f(c) \leq b$ then either f is strictly increasing on $[\tilde{x}, b]$ or strictly decreasing on $[a, \tilde{x}]$. If the first is true then $f(b) > b$ which is not possible by the previous lemma as some critical point of f would then be mapped to some value larger than b. For the same reason f cannot be decreasing on $[a, \tilde{x}]$ since $f(a) > b$ is not possible. Hence, $f(x) \leq b$ on $[a, b]$. A similar argument implies that, $f(x) \geq a$ on $[a, b]$ verifying the claim.

Note that $C \subseteq [a, b]$ where C is the smallest interval containing the critical points of f. Hence, any iterate of C is also in $[a, b]$ as $f([a, b]) \subseteq [a, b]$ and the result follows from the definition of the dynamical interval. □

Corollary 1 *An S^k-multimodal map f can have at most $|\mathcal{C}(f)| + 2$ attracting periodic orbits. If no critical point of f is attracted to a periodic orbit then all periodic points in the critical interval of f and therefore the dynamical interval of f are hyperbolic repelling.*

Proof: The first statement is immediate from Theorem 2.5. To prove the second note that from Lemma 3.3 the critical interval $\tilde{I}^k$ of f^k has the property that its endpoints are either neutral or attracting periodic points that attracts a critical point of f^k or else lie on the orbit of some critical point of f. This together with the second statement of Lemma 3.4 therefore imply that f^k restricted to $\tilde{I}^k$ has the property that each of its attracting periodic orbits attracts a critical point. Hence, if no critical point of f^k is attracted to a periodic orbit, f^k can have no attracting or neutral periodic points in its critical interval.

As each critical point of f^k is a preimage of some critical point of f then the critical interval $\tilde{I}$ of f is contained in $\tilde{I}^k$. Also f^k has a stable, unstable, or neutral periodic orbit through some point if and only if f has a corresponding stable, unstable, or neutral periodic orbit through the same point. This implies that if f^k has no attracting or neutral periodic points in its critical interval then the same is true for f restricted to its critical interval. Furthermore, Lemma 3.4 implies that as the dynamical interval of f is contained in the critical interval $\tilde{I}$ then the dynamical interval of f contains no attracting or neutral periodic points if this is case for $\tilde{I}$. $\square$

Note that it can be shown that the dynamic interval of f is contained in the dynamic interval of f^k but the converse may not be true.

4 Measure Theoretic Properties

In this section we prove Theorems 2.6, 2.7, and 2.8 from section 1, that is those results that are more measure theoretic in nature. We also give some related corollaries that will be used in section 6 in our discussion of the neuronal model.

Lemma 4.1 *Let f be C^1 with critical set $\mathcal{C}(f) = \{c_1, \ldots, c_n\}$ having orders $\ell_1, \ldots, \ell_n$ respectively. The order of a critical point c of f^k is the product of the orders of the critical points of f in the sequence $c, f(c), f^2(c), \ldots, f^{k-1}(c)$.*

Proof: We proceed by induction. Suppose for some fixed $k \in \mathbb{N}$ that $(f^k)'(c) = 0$ and that only one of the points $c, f(c), f^2(c), \ldots, f^{k-1}(c)$ is a critical point of f. Let this be the point $f^j(c)$ for $0 \le j \le k-1$ with order $\ell_j > 1$. Then it follows that

$$\lim_{x\to c} \frac{|(f^k)'(x)|}{|x-c|^{\ell_j - 1}} = \prod_{i=0, i\neq j}^{k-1} f'(f^i(c)) \lim_{x\to c} \frac{f'(f^j(x))}{|x-c|^{\ell_j-1}}. \tag{4}$$

Let $A = \prod_{i=0, i\neq j}^{k-1} f'(f^i(c))$ which is strictly positive and note that

$$\lim_{x\to c} \frac{|f'(f^j(x))|}{|f^j(x) - f^j(c)|^{\ell_j-1}} = \lim_{x\to f^j(c)} \frac{|f'(x)|}{|x - f^j(c)|^{\ell_j-1}} > 0$$

where the inequality follows from definition 2.3(ii). Letting this limit be the quantity B we can rewrite the right side of equation (4) as

$$A \lim_{x\to c} \frac{|f'(f^j(x))|}{|f^j(x) - f^j(c)|^{\ell_j-1}} \frac{|f^j(x) - f^j(c)|^{\ell_j-1}}{|x-c|^{\ell_j-1}} = AB \lim_{x\to c} \Big(\frac{|f^j(x) - f^j(c)|}{|x-c|}\Big)^{\ell_j-1}.$$

An application of L'Hospital's rule implies that this limit is strictly positive since $f^i(c)$ cannot be a critical point of f for $j-1 \leq i \leq 0$. Note that the theorem therefore holds true in the case of one critical point.

Assume now that this holds for m critical points and consider the case in which $(f^k)'(c) = 0$ and $m+1$ of $c, f(c), f^2(c), \ldots, f^{k-1}(c)$ are critical points of f. For these critical points with indices $\alpha_1, \ldots, \alpha_{m+1}$ let, as in the case above, the largest index be $\alpha_{m+1} = j$. Then for $\tilde{\ell} = \prod_{i=1}^{m} \ell_{\alpha_i}$

$$\lim_{x\to c} \frac{|(f^k)'(x)|}{|x-c|^{\ell-1}} = \prod_{i=j+1}^{k-1} f'(f^i(c)) \Big[\lim_{x\to c} \frac{|f'(f^j(x))|}{|x-c|^{\ell-\tilde{\ell}}} \Big] \Big[\lim_{x\to c} \frac{|(f^j)'(x)|}{|x-c|^{\tilde{\ell}-1}} \Big]$$

where by assumption the second limit on the right hand side of this equation is strictly positive. Let $\ell = \ell_j \tilde{\ell}$ and note that

$$\begin{aligned} \lim_{x\to c} \frac{|f'(f^j(x))|}{|x-c|^{\ell-\tilde{\ell}}} &= \lim_{x\to c} \frac{|f'(f^j(x))|}{|f^j(x)-f^j(c)|^{\ell_j-1}} \Big(\frac{|f^j(x)-f^j(c)|}{|x-c|^{\tilde{\ell}}} \Big)^{\ell_j-1} \\ &= B \lim_{x\to c} \Big(\frac{|f^j(x)-f^j(c)|}{|x-c|^{\tilde{\ell}}} \Big)^{\ell_j-1} = B \Big(\lim_{x\to c} \frac{|(f^j)'(x)|}{\tilde{\ell}|x-c|^{\tilde{\ell}-1}} \Big)^{\ell_j-1} > 0 \end{aligned}$$

where the inequality follows again by assumption and the last equality is another use of L'Hospital's rule. The result then follows by induction. □

One immediate consequence of this lemma is that for a function f to be S^k-multimodal or unimodal it is sufficient that $\mathcal{C}(f)$ be a finite set of nonflat critical points and for f to have an eventual negative Schwarzian derivative. On the other hand this is not necessary since the previous lemma also implies that an S^k-multimodal or unimodal function may have flat critical points.

We are now in a position to give the proof of Theorem 2.6.

Proof: In order to apply the main result in [BrLuSt03] to f^k we need to verify that each of its critical points has the same critical order. In fact, it turns out that every critical point has order ℓ. To see this note that for any $\tilde{c} \in \mathcal{C}(f^k)$ in the chain rule expansion of $(f^k)'(\tilde{c}) = \prod_{i=0}^{k-1} f'(f^i(c))$ only one of $c, f(c), f^2(c), \ldots, f^{k-1}(c)$ can be a critical point of f. This follows from the observation that if f eventually maps any point of $\mathcal{C}(f)$ back to this set then condition (2) does not hold. Lemma 4.1 therefore implies that any critical point of f^k has order ℓ.

To show f^k also satisfies the growth condition (2) recall that for any $\tilde{c} \in \mathcal{C}(f^k)$ there exists a unique $m < k$ such that $f^m(\tilde{c}) = c$ where $c \in \mathcal{C}(f)$. For the sake of convenience let

$$\begin{aligned} D_{n,k}(\tilde{c}) &= |((f^k)^n)'(f^k(\tilde{c}))| \\ C_{k,m}(\tilde{c}) &= \prod_{i=1}^{k-m-1} |f'(f^i(\tilde{c}))|, \ m < k. \end{aligned}$$

By simple repeated use of the chain rule it can be shown that

$$C_{k,m}(\tilde{c})D_{n,k}(\tilde{c}) = \prod_{i=1}^{nk-m-1} |f'(f^i(c))|.$$

The important observation here is $C_{k,m}$ does not depend on n and also $D_{nk-m-1}(c) = C_{k,m}(\tilde{c})D_{n,k}(\tilde{c})$ for $n \geq 1$. Since all the quantities involved are positive we have the following bound

$$C_{k,m}^{-1/(2\ell-1)}(\tilde{c}) \sum_n D_{n,k}^{-1/(2\ell-1)}(\tilde{c}) \leq \sum_n D_n^{-1/(2\ell-1)}(c) < \infty. \tag{5}$$

As this is true for all $\tilde{c} \in \mathcal{C}(f)$ then f^k satisfies the required growth condition along the orbits of its critical points.

Since it is entirely possible that f^k has either neutral or attracting periodic orbits in its critical interval $\tilde{I}$ we consider the following. If it is the case that a critical point $c \in \mathcal{C}(f)$ is attracted toward or falls on an attracting periodic orbit of f then it follows that $D_n(f(c)) < C\lambda^n$, where $C > 0$ and $\lambda < 1$ is some number larger than the derivative along the attracting periodic orbit. This implies that for large enough n, $D_n(f(c)) < 1$ or

$$\sum_n D_n^{-1/(2\ell-1)}(c) > C \sum_n (\lambda^{-1/(2\ell-1)})^n = \infty.$$

Assuming (2) implies that this is not the case or no point of the critical set $\mathcal{C}(f)$ can be attracted to a periodic orbit. Corollary 1 therefore implies that all periodic points in the critical interval of f are neither attracting nor neutral.

In the multimodal case it is clear that the critical interval of f has positive Lebesgue measure. Similarly, from the proof of Lemma 3.3 it follows that in the unimodal case the same is true unless in the exceptional case that the critical point is the fixed point of the function. However, in this case (2) cannot hold. Therefore, all functions satisfying condition (2) have a critical interval of positive Lebesgue measure.

As Lemma 3.4 implies the critical interval of f maps into itself then we may consider the map f restricted to its critical interval which we denoted by $\tilde{f}$. From the above the function $\tilde{f} : \tilde{I} \to \tilde{I}$ is C^3 has a finite critical set, has no attracting or neutral fixed points. Therefore, the main result of [BrLuSt03] implies that there exists an $\tilde{f}$-invariant probability measure $\tilde{\mu}$ absolutely continuous with respect to Lebesgue measure acip on $\tilde{I}$. Moreover, some iterate of $\tilde{f}$ is mixing and the correlation function decays with the rates given in Theorem 2.6. The proof follows by considering the measure μ on I equal to $\tilde{\mu}$ on $\tilde{I}$ and zero elsewhere. □

For the proof of Theorem 2.7 we require the following Lemma.

Lemma 4.2 *Let f be C^1. If for some $k \in \mathbb{N}$ there is an f^k-invariant probability measure μ absolutely continuous with respect to Lebesgue measure acip then f*

admits an invariant measure also absolutely continuous with respect to Lebesgue measure.

Proof: Suppose f, k, and μ are as above. Consider the measure υ given by

$$\upsilon(A) = \frac{1}{k}\sum_{i=0}^{n-1} \mu(f^{-i}(A)).$$

To see that υ is f-invariant note

$$\upsilon(f^{-1}(A)) = \frac{1}{k}\sum_{i=1}^{k-1} \mu(f^{-i}(A)) + \frac{\mu(f^{-k}(A))}{k} = \frac{1}{k}\sum_{i=1}^{k-1} \mu(f^{-i}(A)) + \frac{\mu(A)}{k} = \upsilon(A).$$

For absolute continuity of the measure note that a C^1 function with a finite critical set is non-singular. Therefore, absolute continuity of the measure υ follows from that of μ. $\square$

We now give the proof of Theorem 2.7
Proof: The proof of Theorem 2.7 requires that we make two estimates similar to those made in the proof of Theorem 2.6. In fact the only difference is that we would like to replace ℓ by ℓ_{max} and second $-1/(2\ell-1)$ by $-1/\ell_{max}$ in (5). In the first of these note that as in the proof of Theorem 2.6 if $\tilde{c} \in \mathcal{C}(f^k)$ then it must be the preimage of some critical point $c \in \mathcal{C}(f)$ and must also have the same order as c. That is, the collection of orders of $\mathcal{C}(f)$ are the same as those for $\mathcal{C}(f^k)$ so their maximum ℓ_{max} must likewise be equal. From this observation it follows that by replacing $-1/(2\ell-1)$ by $-1/\ell_{max}$ in (5) we obtain the desired inequality for f^k. Hence,

$$\sum_n D_{n,k}^{-1/(\ell_{max})}(c) < \infty \text{ for all } c \in \mathcal{C}(f^k).$$

This implies f^k has an acip from the main result in [NowStr01] and Lemma 4.2 implies the same for f. $\square$

We now proceed to the proof of Theorem 2.8 noting that if condition (i) of Theorem 2.8 holds for $k = 1$ i.e. f_β has a negative Schwarzian derivative then these four conditions in Theorem 2.8 are exactly those used to prove the result in the paper [YoWa06]. Our goal then is to show that we can weaken the hypothesis from a function having a negative Schwarzian derivative to the function having only an eventual negative Schwarzian derivative. The idea for the proof is to first verify that the conditions (i)-(iv) of Theorem 2.8 on f guarantee the function $f_\beta^k(x)$ satisfies conditions (ii), (iii), and (iv) above. It then follows that $f_\beta^k(x)$ has an absolutely continuous invariant measure. With this Theorem 2.8 will follow from an application of Lemma 4.2.
Proof: Firstly, we verify that $\inf_{i>0} d((f_\beta^k)^i(c), \mathcal{C}(f_\beta^k)) > 0$ for all critical points $c \in \mathcal{C}(f_\beta^k)$. This follows as each point in $\mathcal{C}(f_\beta^k)$ is the preimage of some critical

point c of f_β implying that any orbit of the critical set $\mathcal{C}(f_\beta^k)$ must stay some positive distance away from itself as this would otherwise violate condition (iv) on f.

Second we verify that $(f_\beta^k)''(c) \neq 0$ for all critical points $c \in \mathcal{C}(f_\beta^k)$. To do so first note that $(f_\beta)''(c) \neq 0$ implies

$$0 < |(f_\beta'')(c)| = \lim_{x \to c} \frac{|f_\beta'(x)|}{|x-c|^{\ell-1}} \text{ where } \ell = 2$$

or c is a nonflat critical point of f_β by (1). Moreover, as $\inf_{i>0} d((f_\beta^n)^i(c), \mathcal{C}(f_\beta^k)) > 0$ for all critical points $c \in \mathcal{C}(f_\beta^k)$ then no critical point of f is mapped onto the set $\mathcal{C}(f)$ and the same argument used in the proof of Theorem 2.6 implies that all the critical points of f_β^k are nonflat of order $\ell = 2$. Therefore, $(f_\beta^k)''(c) \neq 0$ for all $c \in \mathcal{C}(f_\beta^k)$.

Next, note that property (iii) of Theorem 2.8 means $f_\beta(x)$ has no attracting or neutral periodic orbit. But this is true if and only if $f_\beta^k(x)$ has none itself so this property follows for the function f_β^k.

From this and the assumption that f_β^k has a negative Schwarzian derivative it follows that $f_\beta^k \in \mathcal{M}$. In particular, there is an absolutely continuous measure μ such that $\mu(A) = \mu((f_\beta^k)^{-1}(A))$ for any μ-measurable set A contained in I. An application then of Lemma 4.2 implies that f_β also has an acip. Moreover, if f_β^k satisfies (PT) then on a positive Lebesgue measure set of parameters the family of functions f_a^k has an acip and Lemma 4.2 can again be used to show the same for f_a. $\square$

As we will be concerned specifically with unimodal maps in section 6 we give the following corollary.

Corollary 2 *For the one parameter family of functions $f_a : I \to I$ where a is in some interval of the real line B let $\beta \in B$ such that $f_\beta : I \to I$ is C^3, S^k-unimodal, and satisfies properties (i) and (ii) of Theorem 2.8. If the critical point of f_β is mapped to an unstable periodic orbit then f_β and has an acip on I. Moreover, in this case the critical interval and the dynamical interval of f_β are equal and contain no attracting or neutral periodic orbits. In addition to this if f_β has no fixed points outside this interval then $f_\beta \in \mathcal{M}$.*

Proof: Assuming the conditions above let c be the critical point of f_β. We also assume that f_β is convex since this can always be achieved by an affine change of coordinates. As mentioned in the proof of Lemma 3.3 if $f_\beta(c) \leq c$ then c is attracted to the rightmost fixed point of f_β in I so this is not the case. Then as $f_\beta(c) > c$ if $f_\beta^2(c) \geq c$ the forward orbit of c is contained in the interval $[c, f_\beta(c)]$ on which f_β is strictly decreasing so c is either attracted to a fixed point or a periodic cycle of period 2. Since this is also not possible then $f_\beta^2(c) < c < f_\beta(c)$. As the only critical point of f_β lies between $f_\beta^2(c)$ and $f_\beta(c)$ it follows that both the dynamical interval and critical interval $\tilde{I}$ of f_β are the interval $[f_\beta^2(c), f_\beta(c)]$.

Corollary 1 therefore implies that on $\tilde{I}$ f_β satisfies condition (iii) of Theorem 2.8 since the single critical point c is not attracted to any periodic point. Condition (iv) also follows from the assumption that c maps onto an unstable periodic orbit. Theorem 2.8 then implies that f_β restricted to $\tilde{I}$ has an acip which can be trivially extended to I.

Note that if f_β has no fixed points outside $\tilde{I}$ then there is some $n \in \mathbb{N}$ such that for all $x \in I \setminus \tilde{I}$ $f_\beta^n(x) \in \tilde{I}$ since $\tilde{I}$ is forward invariant and all orbits fall on this set after some bounded number of iterations. Therefore, f_β has no periodic points outside the critical interval so Theorem 2.8 can be used to show $f_\beta \in \mathcal{M}$. $\square$

5 Characterizing Eventually Negative Schwarzian Functions

As not every C^2 function has the characteristics given in Theorem 2.5 it is not the case that every function will be either a function with a negative Schwarzian derivative or have an eventual negative Schwarzian derivative. What is more likely the case is that a function will have a Schwarzian derivative that is mixed for all of its iterates. If however a function does have an eventual negative Schwarzian derivative we would like to have some way of identifying this. More precisely, we would like to give sufficient conditions under which a function is eventually Schwarzian.

Proposition 1 *Let $f : I \to I$ be C^2. Suppose $L \subseteq I$ is an open set on which f has nonzero constant slope and that on $I \setminus \bar{L}$ f has a negative Schwarzian derivative. Furthermore, assume there is a $k \in \mathbb{N}$ such that for every $x \in L$ $\{f^i(x) : 1 \leq i \leq k\} \cap I \setminus \bar{L} \neq \emptyset$. Then $f(x)$ has an eventual negative Schwarzian derivative on I of order less than or equal to k.*

Proof: We first note that if f has a negative Schwarzian derivative in some neighborhood of $ax+b$, $a \neq 0$ then $|(f(ax+b))'|^{-1/2} = a^{-1/2}|f'(ax+b)|^{-1/2}$ is strictly convex in this neighborhood. Similarly, if f has a negative Schwarzian derivative in some neighborhood of x then $|(af(x)+b)'|^{-1/2} = a^{-1/2}|f'(x)|^{-1/2}$ is strictly convex in this neighborhood.

Since $f^k = f \circ f \circ \cdots \circ f$ is the composition of such functions where for any $x \in I$ there is an $i < k$ such that $f(x)$ has a negative Schwarzian derivative at $f^i(x)$ then the result follows from the observation above and the fact that the property of having a negative Schwarzian derivative is preserved under composition. $\square$

It follows directly from this proposition that the C^2 family of unimodal functions

$$f_a(x) = \begin{cases} (x - 1/2) + a & x \in [0, 1/2] \\ -4(2a+1)(x-1/2)^3 + (x - 1/2) + a & x \in (1/2, 1] \end{cases} \tag{6}$$

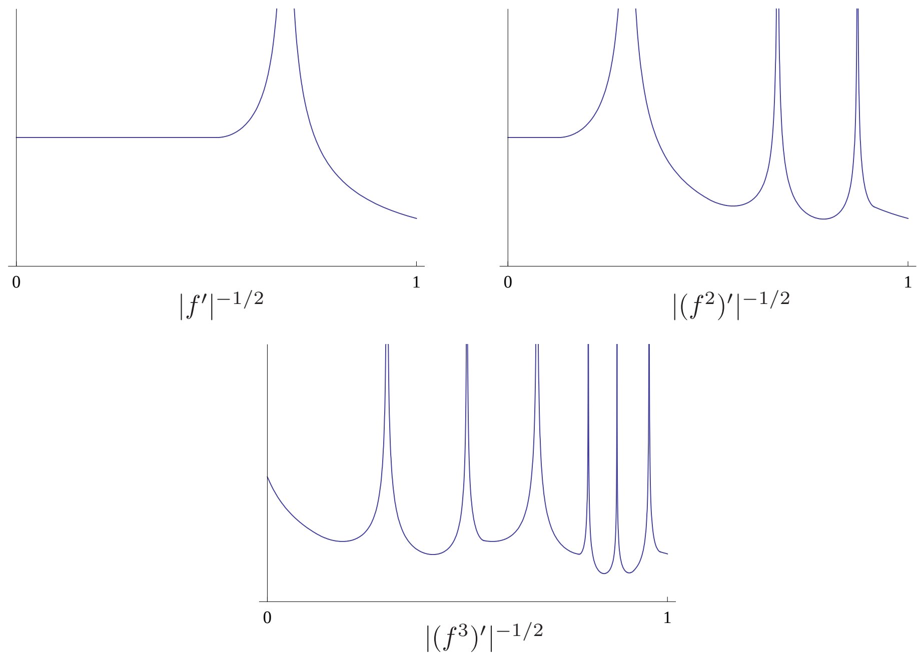

Figure 1: The strict convexity of $|(f^3)'|^{-1/2}$ implies $f = f_{7/8}$ given in (6) has an eventual negative Schwarzian derivative.

has an eventual negative Schwarzian derivative for $a \in (1/2, 7/8]$ (see Fig. 1). Note that this family of functions is an example of a collection of unimodal maps which are not guaranteed via the results of Kozlovski or Graczyk, Sands, and Swiatek [Koz00, vSV04] to have some first return map with a negative Schwarzian derivative but nevertheless has this property.

It is also worthwile to recall, as mentioned in the introduction, the incentive for introducing this new class of functions with eventual negative Schwarzian derivatives comes from the analysis of one-dimensional maps which appear in some problems in neuroscience. These maps [Med04, Med06] in particular have a part that is linear (or almost linear). Therefore, our example above also includes such a part although this is not a necessary condition to have an eventual negative Schwarzian derivative (see e.g. (9)).

It is well known that if a C^3 function f has a negative Schwarzian derivative on some interval then this is equivalent to $S(f(x)) < 0$ where

$$S(f(x)) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\Big(\frac{f''(x)}{f'(x)}\Big)^2 \tag{7}$$

where $S(f(x))$ can be $-\infty$ at a finite number of points.

Remark 1 *If $f : I \to I$ is C^3 with finite critical set and $S(f(x)) < 0$ on the* *interval I then, as $S(f(x))$ is discontinuous only at the finite number of points where it is equal to $-\infty$, continuity elsewhere implies that there is some $\lambda > 0$ such that $S(f(x)) < -\lambda < 0$ for all $x \in I$. Moreover, as $S(f(x))$ is a continuous function of $f'(x)$, $f''(x)$, and $f'''(x)$ then for $\epsilon > 0$ small enough, if $d_3(f,g) < \epsilon$ then g also has a negative Schwarzian derivative with this C^3-distance defined by*

$$d_3(f,g) = \sup_{x\in I}\{|f'(x) - g'(x)|, |f''(x) - g''(x)|, |f'''(x) - g'''(x)|\}.$$

That is, the property of having a negative Schwarzian derivative is maintained under small C^3 perturbations implying the same for functions with eventual negative Schwarzian derivatives.

Returning to (7) it can be shown that

$$S(f^n(x)) = \sum_{i=0}^{n-1} \Big((f^i)'(f^i(x))\Big)^2 \Big(S(f(f^i(x)))\Big). \tag{8}$$

One way to interpret this is that the Schwarzian derivative of $f^k(x)$ is the weighted sum of $S(f(x))$ along the orbit of x with the weight being the square of the derivatives also calculated along this orbit. This interpretation has the advantage that it suggests a method for identifying functions which eventually have negative Schwarzian derivatives. This is the main idea behind the following theorem which is similar to the previous proposition.

Theorem 5.1 *Let $f : I \to I$ be C^3 where $M = \max_{x\in I}\{|f'(x)|\}$, I is the union of two disjoint sets L and $I \setminus L$, and the following conditions hold:*
i) There is a $\lambda > 0$ such that $S(f(x)) < -\lambda$ for all $x \in I \setminus L$.
ii) There is a $m > 0$ such that for every $x \in L$ $|f'(x)| > m$.
iii) There is a $k \in \mathbb{N}$ such that for every $x \in L$ $\{f^i(x) : 1 \le i \le k\} \cap I \setminus L \neq \emptyset$.
Let $\Lambda = \max_{x\in L}\{f'''(x)/m - \frac{3}{2}(f''(x)/m)^2\}$. Then f has an eventual negative Schwarzian derivative of order less than or equal to k if either

$$\Lambda\tfrac{1-M^{2k}}{1-M^2} < \lambda m^{2k-2} \text{ when } m < 1 \text{ or if } \Lambda\tfrac{1-M^{2k}}{1-M^2} < \lambda \text{ when } m \ge 1.$$

Proof: Suppose the conditions of the theorem are satisfied. Then for any $x \in I$ ii) implies for some $0 \le i \le k-1$ that $f^i(x) \in I \setminus L$. From equation (8) we can make the following estimation

$$S(f^k(x)) < \Lambda + (M^2)\Lambda + \cdots - (m^{2i})\lambda + \cdots + (M^2)^{k-1}\Lambda.$$

Adding $(M^2)^i\Lambda$ to the right hand side of the inequality we obtain

$$S(f^k(x)) < \Lambda\sum_{m=0}^{k-1}(M^2)^m - \lambda m^i = \Lambda\Big(\frac{1-M^{2k}}{1-M^2}\Big) - \lambda m^i$$

which is nonpositive if $\Lambda\Big(\frac{1-M^{2k}}{1-M^2}\Big) < \lambda m^{2i}$. If it is the case that $m < 1$ then the theorem above follows from the fact that $m^{2(k-1)} < m^{2i}$. On the other hand if

$m \geq 1$ then $m^{2i} \geq 1$. This completes the proof. □

It should be clear that in using Theorem 5.1 we are neglecting much of the possible structure a function may have e.g. how long orbits stay in $I \backslash L$, possible return times to this set, and other such dynamic information that could be used to sort out whether or not a function has an eventual negative Schwarzian derivative. It would be interesting to know if there are other useful criteria other than that given in the previous theorem under which this is also the case.

As another example the one parameter family of functions

$$g_a(x) = 1 - a \tan\left(\frac{\pi}{4}x^2\right),\ x \in [-1, 1],\ a \in [1, 3/2] \tag{9}$$

has an eventual negative Schwarzian derivative of order 2. In contrast to the previous example this family of functions is C^3 with non-flat critical point. In this case it follows from the result by Kozlovski [Koz00] that this function does have a first return map around its critical point having a negative Schwarzian derivative. However, more than this, for example we have from Corollary 1 that g_a has at most 3 attracting periodic orbits which is based not on the fact that g_a is C^3 with non-flat critical points but that it has an eventual negative Schwarzian derivative.

6 Application to a Neuronal Model

The motivation for considering functions having some iterate with a negative Schwarzian derivative came from a model for the electrical activity in neural cells specifically in behavior described as bursting. This model given first in [Med04] and later in [Med06] is a reduction of a system of three nonlinear differential equations to a 1-d map.

Initially the model is given by the following system of three differential equations which describe the dynamics of the membrane potential υ and two gating variables η and ω of a neuronal cell:

$$\alpha\dot{\upsilon} = f(\upsilon, \eta, \omega; \delta) \tag{10}$$
$$\dot{\eta} = g(\eta, \upsilon) \tag{11}$$
$$\dot{\omega} = \beta h(\eta, \omega) \tag{12}$$

Here the parameter δ can be viewed as a control parameter of the full system where $\delta \in [0, \delta_{max}]$. Also the time constant β represents the slowest time scale in the dynamics of (10)-(12) so in the limit $\beta \to 0^+$ the system uncouples into a fast subsystem (10), (11) and a slow subsystem (12). As is explained in [Med06] the trajectory of the full system is drawn towards a surface foliated by periodic orbits of the fast subsystem where the evolution along this surface is determined by the dynamics of the slow subsystem. As the state of the fast subsystem depends on the value of the slow variable ω, it is sufficient to know

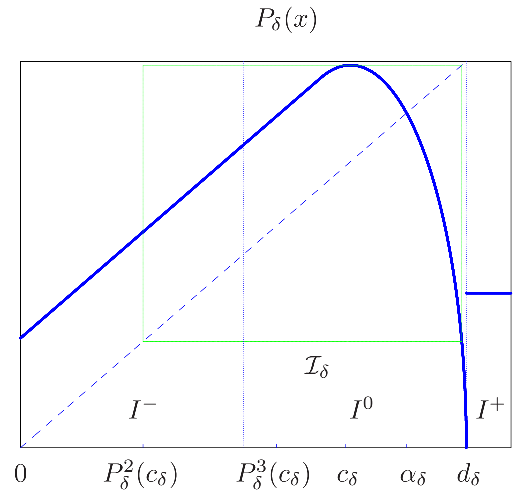

Figure 2: First Return Map Near Bursting $\delta \approx \delta_b$

how ω changes after each oscillation of the fast subsystem. Knowing these changes is precisely the reduction of the system to a 1-d map denoted by P_δ.

In [Med04] P_δ is shown to have the following properties (see Fig. 2):

(i) P_δ is a piecewise continuous map with two intervals of continuity $I_1 = I^- \cup I^0$ and $I_2 = I^+$ between which there is a single discontinuity d_δ.

(ii) $P_\delta(x)$ is unimodal on $0 \leq x \leq d_\delta = I_1$, with vertex c_δ at which $P_\delta(x)$ is to leading order quadratic, i.e. nonflat.

(iii) P_δ is nearly linear on its left outer region I^- with slope slightly less than 1. On the inner region I^0 P_δ is unimodal with slope tending toward $-\infty$ as d_δ is approached from the left.

(iv) In the second region I^+ the function is nearly constant and between 0 and $P_\delta(c_\delta)$.

(v) Away from d_δ, $P_\delta(x)$ is pointwise continuous in δ following from continuous dependence on parameters of the unreduced system.

(vi) There exists a $\delta_0 \in (0, \delta_{max})$ such that $\alpha_{\delta_0} = c_{\delta_0}$ and for $\delta \in (\delta_0, \delta_{max}]$, $c_\delta < \alpha_\delta$ as well as a $\delta_b \in [\delta_0, \delta_{max})$ such that $\delta_0 < \delta_b$, $P_{\delta_b}(c_{\delta_b}) = d_{\delta_b}$. Also for $\delta \in [0, \delta_b)$, $P(c_\delta) < d_\delta$.

(vii) P_δ has a unique fixed point α_δ on the interval I^0.

Our aim in this section is to give sufficient conditions under which this one parameter family of functions has an acip for a positive Lebesgue measure set of its parameters. Also under what conditions these functions exhibit a mixing property with respect to these measures.

It will be this parameter interval $[\delta_0, \delta_b)$ given in (vi) which we will concentrate on. The reason being is that for parameter values prior to δ_0 the map P_δ has a global attracting fixed point so the dynamics are easily understood. And for parameters larger than δ_b the map P_δ is not a continuous map so the theory does not apply. Specifically, we make the observation that so long as $\delta \in [\delta_0, \delta_b)$ then $P_\delta(c_\delta) < d_\delta$ implying $P_\delta(I_1) \subseteq I_1$. Since $P_\delta(I_2) \subseteq I_1$ then for such parameter values it suffices to consider the map restricted to I_1 on which the map is unimodal.

If $\delta \in [\delta_0, \delta_b)$ then we denote by $\mathcal{I}_\delta$ the critical interval of P_δ. Since in this parameter range $c < f(c)$ then $\mathcal{I}_\delta = [c, f(c)]$ if $c \leq f^2(c)$ or if $f^2(c) < c$ then $\mathcal{I}_\delta = [f^2(c), f(c)]$. If the latter is the case then this is also the dynamical interval of f.

We will need the following technical lemma most of which is meant to establish basic continuity properties of the family of functions $P_\delta(x)$ under variation of parameters.

Lemma 6.1 *For all $\delta \in [\delta_0, \delta_b)$ let P_δ be C^2 such that*
(i) c_δ and P'_δ are continuous in the parameter δ,
(ii) $P_\delta(x)$ is concave down on the interval $[c_\delta, P_\delta(c_\delta)]$,
(iii) there is some $\gamma \in (\delta_0, \delta_b)$ such that both $P^2_\gamma(c_\gamma) < c_\gamma$ and $P^3_\gamma(c_\gamma) < c_\gamma$.
Then there exists an infinite subset $\Delta \subseteq (\delta_0, \delta_b)$ such that for every $\delta \in \Delta$ the forward orbit of the critical point c_δ falls on the fixed point α_δ.

Proof: From continuity of P'_δ and c_δ in the parameter it follows that

$$\lim_{\delta \to \delta_0^+} P'_\delta(P_\delta(c_\delta)) = P'_{\delta_0}(P_{\delta_0}(c_{\delta_0})) = P'_{\delta_0}(c_{\delta_0}) = 0.$$

As $P'_\delta(P_\delta(c_\delta)) < 0$ for all $\delta \in (\delta_0, \delta_b)$ then for some $\delta_1 \in (\delta_0, \delta_b)$ on the interval (δ_0, δ_1) we have $-1 < P'_\delta(P_\delta(c_\delta)) < 0$. Since $P_\delta(x)$ is assumed concave down on the interval $[c_\delta, P_\delta(c_\delta)]$ it follows that $|P^2_\delta(c_\delta) - P_\delta(c_\delta)| < |c_\delta - P_\delta(c_\delta)|$. In particular this implies that $c_\delta < P^2_\delta(c_\delta)$ on the interval of parameters (δ_0, δ_1) and therefore $\delta_1 < \gamma$.

Since P_δ is decreasing on $[c_\delta, P_\delta(c_\delta)]$ if $c_\delta < P^2_\delta(c_\delta)$ as is the case for $\delta \in (\delta_0, \delta_1)$ it follows that $\alpha_\delta < P^3_\delta(c_\delta)$. On the otherhand, since at the parameter value γ, $P^3_\gamma(c_\gamma) < c_\gamma < \alpha_\gamma$ then by continuity there exists $\tilde{\delta} \in (\delta_1, \gamma)$ such that $P^3_{\tilde{\delta}}(c_{\tilde{\delta}}) = \alpha_{\tilde{\delta}}$. Similarly, as $P^3_{\tilde{\delta}}(c_{\tilde{\delta}}) > c_{\tilde{\delta}}$ and $P^3_\gamma(c_\gamma) > c_\gamma$ then by continuity there is some $\delta_2 \in (\tilde{\delta}, \gamma)$ such that $P^3_{\delta_2}(c_{\delta_2}) = \alpha_{\delta_2}$. What is important here is this implies that for $\delta \in [\tilde{\delta}, \delta_2]$ the value of $P^3_\delta(c_\delta)$ ranges continuously from $\alpha_{\tilde{\delta}}$ to c_{δ_2}.

Now note that if for some $\delta \in (\delta_0, \delta_b)$ it happens that $P^3_\delta(c_\delta) < \alpha_\delta$ then every point in $(\alpha_\delta, P_\delta(c_\delta))$ has exactly two preimages; one preimage in $l(\delta) := (P^2_\delta(c_\delta), c_\delta)$ and the other in $r(\delta) := (c_\delta, \alpha_\delta)$. As every point in $l(\delta)$ and $r(\delta)$ has a unique preimage in $s(\delta) := (\alpha_\delta, P_\delta(c_\delta))$ we can specify a preimage of α_δ by some finite sequence made up of l, r, and s which stand for whether the point is reached from the fixed point by tracing the path backward through these sets. Note that we define a finite sequence of l, r, and s to be admissible if and only if an s separates every l or r in the sequence. This corresponds to the structure above and uniquely defines a preimage of α_δ.

As there are infinitely many sequences which can end in either l, r, s then there exist infinitely (uncountably) many preimages of the fixed point α_δ in each of $l(\delta)$, $r(\delta)$, and $s(\delta)$. Also since a preimage is defined by a specific zero of the function $P_\delta^n(x) - x$ then these preimages are continuous in the parameter δ. As the last letter in the sequence of a preimage determines whether it is in $l(\delta), r(\delta)$, or $s(\delta)$ it must stay in that interval for all δ such that $P_\delta^3(c_\delta) < \alpha_\delta$.

This along with the observation above that $P_\delta^3(c_\delta)$ continuously ranges from $\alpha_{\tilde{\delta}}$ to c_{δ_2}, that is over $r(\delta)$, implies the existence of the infinite set $\Delta \subseteq (\delta_0, \delta_b)$ on which the orbit of the critical point c_δ is mapped to the fixed point c_δ. $\square$

Theorem 6.2 *For all $\delta \in [\delta_0, \delta_b)$ let P_δ be C^3 on its critical interval such that the following conditions hold:*
(i) c_δ and P'_δ are continuous in the parameter δ,
(ii) $P_\delta(x)$ is concave down on the interval $[c_\delta, P_\delta(c_\delta)]$,
(iii) there is some $\gamma \in (\delta_0, \delta_b)$ such that both $P_\gamma^2(c_\gamma) < c_\gamma$ and $P_\gamma^3(c_\gamma) < c_\gamma$.
Consequently, there is an infinite set of parameters $\Delta \in (\delta_0, \delta_b)$ such that for $\delta \in \Delta$ if (iv) $|P'_\delta(\alpha_\delta)| > 1$ and (v) P_δ is S^k-unimodal then the following is true:

(A) $P_\delta \in \mathcal{M}$ and therefore has an acip. If in addition the function P_δ^k has property (PT) then on a positive Lebesgue measure set of parameters P_δ has an acip.

(B) P_δ is mixing with exponential decay of correlations.

Proof: For $\delta \in (\delta_0, \delta_b)$ let $[a_\delta, b_\delta]$ be the critical interval of P_δ and consider the function $h_\delta(x) = (b_\delta - a_\delta)x + a_\delta$. Conjugating the family of functions P_δ by h_δ is simply an affine change of coordinates that produces a new one parameter family of maps $\tilde{P}_\delta$ with unique fixed point $\tilde{\alpha}_\delta$ and critical point $\tilde{c}_\delta$. What is important here is that when restricted to the interval $[0, 1]$ this new family of functions is equivalent to the family P_δ resticed to their critical intervals.

From the previous lemma it follows that there is an infinite set of parameters $\Delta \in (\delta_0, \delta_b)$ such that for all $\delta \in \Delta$ the orbit of c_δ contains the fixed point α_δ. As the same is true then of the family $\tilde{P}_\delta$ it follows immediately from Corollary 2 that this family has property (A). For part (B) note that for $\delta \in \Delta$ there is a $C > 0$ such that $D^n(\tilde{P}_\delta(\tilde{c}_\delta)) > C|\tilde{P}'_\delta(\tilde{\alpha}_\delta)|^n$ as the orbit of c_δ contains a repelling fixed point. By assumption $|\tilde{P}'_\delta(\alpha_\delta)| = \lambda > 1$. Hence, $D^n(\tilde{P}(\tilde{c}_\delta)) > C\lambda^n$. As $\tilde{P}_\delta$ is nonflat at $\tilde{c}_\delta$ then the order of this critical point is some $\ell \in (1, \infty)$ and we have the following estimate:

$$\sum_n D_n^{-1/(2\ell-1)}(c_\delta) < C \sum_n (\lambda^{-1/(2\ell-1)})^n = \frac{C}{1-\lambda^{1/(2\ell-1)}} < \infty.$$

Part (B) follows for the family $\tilde{P}_\delta$ from an application of Theorem 2.6. Since properties (A) and (B) are maintained under the change of coordinates induced by h_δ both hold for P_δ by extending the relevant measures in the trivial way to the domain of P_δ. □

Conditions (i)-(v) in Theorem 6.2 may seem somewhat arbitrary. However, for P_δ there are reasons to expect that these conditions are satisfied. Specifically, for condition (iii) numerical simulations indicate that this is the case particularly for γ near δ_b (see Fig.2 a replication of Fig.3 [Med06]). Similarly, as the fixed point α_δ is shown in [Med04] to lose stability prior to $\delta = \delta_b$ condition (iv) is likely to be satisfied on some if not all of Δ. Also, as is shown in this paper, for condition (v), as $\beta \to 0$ then on I^- P_δ becomes truly linear in the limit. Proposition 1 along with Remark 1 then imply that for β near 0 we need only check to see if the Schwarzian derivative is negative on the set $\mathcal{I}_\delta \setminus I^-$. If this is not the case then Theorem 5.1 may be considered as an alternative to show P_δ has an eventual negative Schwarzian derivative.

It should be noted that functions with the property that their critical points are mapped to repelling periodic orbits, often called *Misiurewicz maps*, are very rare both in a topological and metric sense [San98]. From this point of view part (B) of the previous theorem seems to imply that mixing is a rare event for the family of functions P_δ. However, the conditions of Theorem 2.6 are generically satisfied on a larger set of parameters than just Δ. The problem which is generally encountered is in showing for a particular parameter value or set of values that the derivatives along the orbits of the critical points have the requisite growth. However, one reason to expect that (B) holds for a much larger set of parameters than just Δ is the very large negative slope of P_δ near d_δ, for $\delta \approx \delta_b$.

7 Conclusion

The goal of this paper was first and foremost to study a new class of functions that behave in some significant ways like those with a negative Schwarzian derivative. For a given function it may be very nontrivial to determine whether or not the function has such a property. However, in a few simple cases we do have a quick way of determining if a function has an eventual negative Schwarzian derivative but this is for functions with a relatively simple structure (see Proposition 1 and Theorem 5.1). However, once it has been determined that a function does have an eventual negative Schwarzian derivative then the function possesses many of the nice properties that functions with negative Schwarzian derivatives have.

It would be interesting to know which properties of functions with a negative Schwarzian derivative do and do not extend to this larger class of functions. In fact the main results of this paper are simply the first step in this direction.

8 Acknowledgments

The author would like to thank L. Bunimovich for suggesting the study of this problem and for his useful comments and discussion.

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