On a Converse of Sharkovsky's Theorem

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In 1964, the Ukranian Mathematician Alexander Nikolaevich Sharkovsky [5] discovered a spectacular result on continuous maps on intervals. For the convenience of the reader we will state Sharkovsky's Theorem in which he used the following ordering of the set of natural numbers:

$3 \rhd 5 \rhd 7 \rhd \ldots$	$2 \times 3 \rhd 2 \times 5 \rhd 2 \times 7 \rhd \ldots$	$2^2 \times 3 \rhd 2^2 \times 5 \rhd 2^2 \times 7 \rhd \ldots$
odd integers	$2 \times$ odd integers	$2^2 \times$ odd integers
$\ldots$	$\ldots$	$\ldots \rhd 2^3 \rhd 2^2 \rhd 2 \rhd 1$

Here $m \rhd n$ signifies that m appears before n in the Sharkovsky ordering.

Theorem 1. (Sharkovsky [5]). *Let f: $I \to I$ be a continuous map from the interval I into itself. If $k \rhd r$ and f has a point of period k, then f must have a point of period r.*

The question that we are going to address in this note is the following: Given any positive integers k, r with $k \rhd r$, is there a continuous map that has a point of period r but no points of period k?

There are very few examples in the literature, which is scattered in many books such as [2–4]. These examples deal mostly with maps that have points of period 5 but no points of period 3 and no pattern is given to generate more examples. Moreover, examples of maps that have points of period 2^n seem missing in textbooks on dynamical systems. However, in an article by Štefan [6], a general scheme was given to generate maps that have points of period $(2n + 1)$ but no points of period $(2n - 1)$. Furthermore, using the so-called "doubling" of maps, he was able to construct maps that have points of period $2^k(2n + 1)$ but no points of period $2^k(2n - 1)$ for any positive integer n and any nonnegative integer k. Clearly, using this scheme one can generate maps that have points of period 2^k but no points of period 2^{k+1}. In this note, however, we give new and simple constructions for such maps. In addition, our proofs are very simple and should be accessible to nonspecialists. We are now ready to state our main result, which we call the Converse of Sharkovsky's Theorem.

Theorem 2. *For any positive integer r there exists a continuous map f_r: $I_r \to I_r$ on the interval I_r such that f_r has points of prime period r but no points of prime period s for all positive integers s that precede r in the Sharkovsky ordering, i.e., $s \rhd \cdots \rhd r$.*

Proof: The proof will be accomplished by the construction of the required maps. Here we have three cases to consider:

(I) odd periods,
(II) periods of the form $2^n \times$ odd natural number,
(III) periods that are powers of 2, i.e., 2^n.

Case I. Odd Periods

(a) A map that has points of period 5 but no points of period 3. Define a map f: $[1,5] \to [1,5]$ as follows:

Let $f(1) = 3$, $f(2) = 5$, $f(3) = 4$, $f(4) = 2$, $f(5) = 1$ and on each interval $[n, n+1]$ we assume f to be linear (see Figure 1).

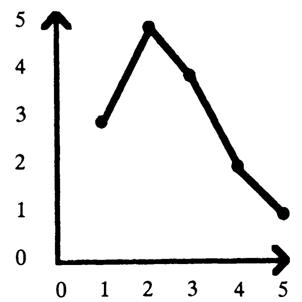

Figure 1

Observe first that none of the points 1, 2, 3, 4, 5 is a periodic point of period 3; they all belong to a 5-cycle. Notice also that

$$f^3([1,2]) = [2,5], \quad f^3([2,3]) = [3,5], \quad \text{and} \quad f^3([4,5]) = [1,4].$$

From these observations we conclude that the third iterate f^3 has no fixed points in the intervals $[1,2]$, $[2,3]$, and $[4,5]$. The situation with the interval $[3,4]$ is, however, more involved, since $f^3([3,4]) = [1,5]$. Then there are points $a, b \in [3,4]$ such that $f^3(a) = 3$, $f^3(b) = 4$. Define a map h: $[1,5] \to R$ by letting $h(x) = x - f^3(x)$. Then $h(a) \geq 0$, $h(b) \leq 0$. Hence by the intermediate value theorem, there exists a point $p \in [3,4]$ with $h(p) = 0$ or $f^3(p) = p$. We will show that p is unique and is a fixed point of f. Now $f(p) \in [2,4]$. So if $f(p) \in [2,3]$, then $f^2(p) \in [4,5]$ and thus $p = f^3(p) \in [1,2]$, which is false. Thus $f(p) \in [3,4]$ and consequently $f^2(p) \in [2,4]$. Again if $f^2(p) \in [2,3]$, then $p \doteq f^3(p) \in [4,5]$, yet another contradiction. Therefore, p, $f(p)$, and $f^2(p)$ all belong to the interval $[3,4]$. Now on the interval $[3,4]$, $f(x) = 10 - 2x$ has the unique fixed point $x^* = 10/3$. Moreover, on $[3,4]$, $f^3(x) = 30 - 8x$, which has the unique fixed point $x^* = 10/3$. Thus $p = x^* = 10/3$, and consequently f has no points of period 3.

(b) Now one can generalize this construction to manufacture continuous maps that have points of period $2n+1$ but no points of period $2n-1$ as follows:

Let f: $[1, 2n+1] \to [1, 2n+1]$ be defined by putting $f(1) = n+1$, $f(2) = 2n+1$, $f(3) = 2n$, $f(4) = 2n-1, \dots, f(n) = n+3$, $f(n+1) = n+2$, $f(n+2) = n$, $f(n+3) = n-1, \dots, f(2n+1) = 1$ (see Figure 2).

First we observe that all the integers in the interval $[1, 2n+1]$ are of period $2n+1$. For example, the orbit of the point 1 is given by the string

$$1 \xrightarrow{f} n+1 \xrightarrow{f} n \xrightarrow{f} n+2 \xrightarrow{f} n-1 \xrightarrow{f} n+3 \xrightarrow{f} n-2 \xrightarrow{f} \cdots \xrightarrow{f} 2 \xrightarrow{f} 2n+1.$$

Observe that, in addition to 1, there are two sequences of length n; one increasing: $\{n+2, n+3, \dots, 2n+1\}$; and another decreasing: $\{n+1, n, \dots, 2\}$. It remains to show that there are no points of period $2n-1$ in the interval $[1, 2n+1]$. Let us

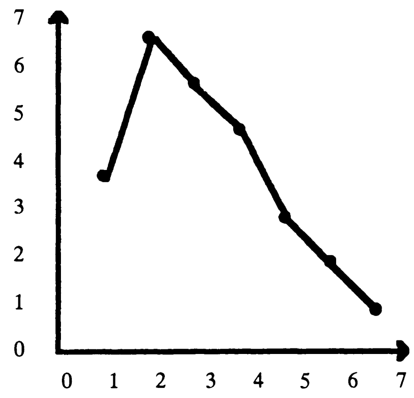

Figure 2

start with the interval $[1,2]$. Now $(2n-1)$ iterations of the interval $[1,2]$ give rise to the following string:

$$[1,2] \xrightarrow{f} [n+1,2n+1] \xrightarrow{f} [1,n+2] \xrightarrow{f} [n,2n+1] \xrightarrow{f} [1,n+3] \xrightarrow{f}$$
$$[n-1,2n+1] \xrightarrow{f} \cdots \xrightarrow{f} [1,2n] \xrightarrow{f} [2,2n+1].$$

This shows that $f^{2n-1}([1,2]) \cap [1,2) = \phi$. Hence the interval $[1,2]$ contains no points of period $(2n-1)$. Now, we can show that all the intervals $[j,j+1]$, with the exception of the interval $[n+1,n+2]$, display the same behavior as that of the interval $[1,2]$. In particular, we can show that there exists an iterate of the interval $[j,j+1]$ that is precisely the interval $[1,2]$. Since the interval $[1,2]$ has no points of period $(2n-1)$, it follows that the interval $[j,j+1]$ has no points of period $(2n-1)$. As for the interval $[n+1,n+2]$, notice that $f[n+1,n+2] = [n,n+2]$. Hence there are two cases for $x \in [n+1,n+2]$.

Case (a) $f^k(x) \in [n+1,n+2]$ for all $k \in Z^+$. Since $|f'| > 1$ on the interval $[n+1,n+2]$, it follows that x is actually a fixed point of f.

Case (b) $f^k(x) \notin [n+1,n+2]$ for some positive integer k. Then $f^k(x) \in [n,n+1]$ and by the previous analysis an iterate of x lies in the interval $[1,2]$.

In either case, x cannot have period $(2n-1)$.

Case II. Maps that have points of period $2^k(2n+1)$ but no points of period $2^k(2n-1)$.

Let us start with period 2×5 but not 2×3. We consider first the map $f\colon I \to I$, $I = [1,5]$ which was considered in Case Ia (Figure 1).

Define a new map $g\colon [1,13] \to [1,13]$ as follows:

$$g(x) = \begin{cases} f(x) + 8, & \text{for } 1 \le x \le 5 \\ x - 8, & \text{for } 9 \le x \le 13 \end{cases}$$

and for $5 < x < 9$, we connect the points $(5,9)$ and $(9,1)$ by a line (see Figure 3).

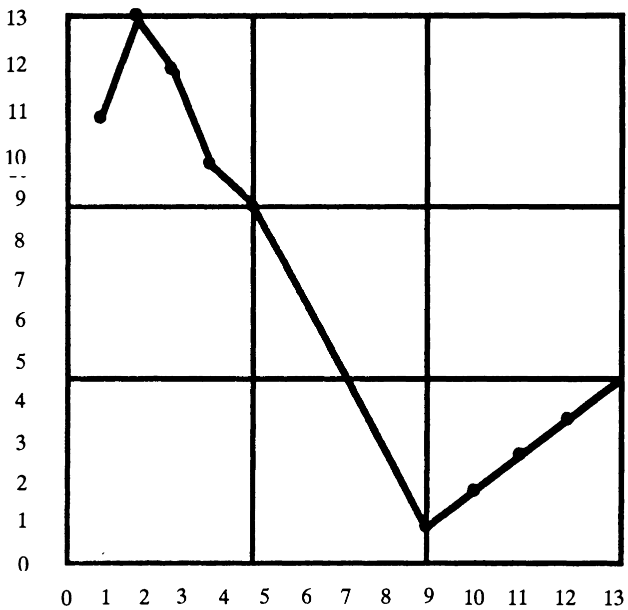

Figure 3

The map g is called the "double of f." Observe first that none of the points $1, 2, 3, 4, 5, 9, 10, 11, 12, 13$ is of period 6; they all belong to a 10-cycle. Moreover, if $x \in [1, 5]$, then $g(x) \in [9, 13]$ and $g^2(x) = f(x)$. Since f has no points of period 3, it follows that g has no points of period 6 in the interval $[1, 5]$. Since $g[9, 13] = [1, 5]$, it follows that no point in the interval $[9, 13]$ is of period 6. The situation with the interval $[5, 9]$ requires a different argument. Since $g^6[5, 9] = [4, 10]$, it follows by an argument similar to that used in Case I(a) that g^6 has a fixed point $p \in (5, 9)$. Now for any n, $1 \le n \le 5$, $g^n(p) \notin (5, 9)$, then $g^{n+r}(p) \in [1, 5] \cup [9, 13]$ for all $r > 0$. This implies that $g^6(p) \neq p$, a contradiction. Thus $p, g(p), \ldots, g^5(p) \in (5, 9)$. By simple computations, one can show that the only fixed point of $g, g^2, \ldots, g^6$ is $p = 19/3$. Thus, g has no points of period 6.

The general procedure for constructing a map that has points of period $2(2n + 1)$ but no points of period $2(2n - 1)$, $n = 1, 2, 3, \ldots$ may be explained as follows. We start with a map f: $[1, 1 + h] \to [1, 1 + h]$ with points of period $(2n + 1)$ but no points of period $(2n - 1)$. We define the double map g: $[1, 1 + 3h] \to [1, 1 + 3h]$ as follows:

$$g(x) = \begin{cases} f(x) + 2h, & \text{for } 1 \le x \le 1 + h \\ x - 2h, & \text{for } 1 + 2h \le x \le 1 + 3h \end{cases}$$

and by linearity for $1 + h < x < 1 + 2h$. Repetition of the preceding scheme would create maps with points of period $2^k(2n + 1)$ but no points of period $2^k(2n - 1)$, $k = 2, 3, 4, \ldots$

Case III. Periods of the form 2^n

(a) A map that has points of period 2 but no points of period 2^2. Let f: $[1, 2] \to [1, 2]$ be defined by $f(x) = -x + 3$. Here every point, except the fixed point $3/2$, in the interval $[1, 2]$ is of prime period 2. Hence there are no points in the interval $[1, 2]$ with prime period 2^2.

(b) A map that has points of period 2^2 but no points of period 2^3. Let f: $[1,4] \to [1,4]$ be defined as follows: $f(1) = 3$, $f(2) = 4$, $f(3) = 2$, $f(4) = 1$ and on each interval $[n, n+1]$ we assume f to be linear (Figure 4).

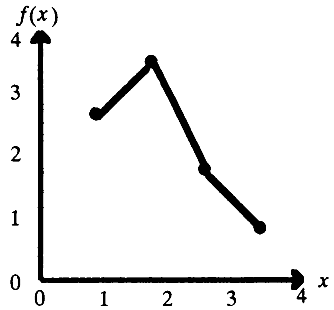

Figure 4

Notice that $f([1,2]) = [3,4]$ and $f([3,4]) = [1,2]$, and f is linear on $[1,2]$ and $[3,4]$. Thus $f^2([1,2]) = [1,2]$ and $f^2([3,4]) = [3,4]$. Also, f^2 is decreasing, as f^4 is increasing. Therefore $f^4(x) = x$ for all $x \in [1,2] \cup [3,4]$.

Hence every point in the interval $[1,2]$ is of prime period 4 except the point $3/2$, which is of prime period 2. Similarly, every point in the interval $[3,4]$ is of prime period 4 except the point $7/2$, which is of prime period 2. Next we deal with the interval $[2,3]$. Since $f[2,3] = [2,4]$, points in the interval $[2,3]$ either leave the interval $[2,3]$ after many iterations or stay in the interval $[2,3]$ for all iterations. Now if for a point $x \in [2,3]$, and for some $k \in Z^+$, $f^k(x) \in [1,2] \cup [3,4]$, then its orbit will be attracted to either a 4-cycle or a 2-cycle. On the other hand if the orbit of $x \in [2,3]$ is a subset of the interval $[2,3]$, then $f^n(x) = f_2^n(x)$, where $f_2(x) = -2x + 8$. But $f_2^8(x) = 256x - 680$ has the fixed point $x^* = 8/3$, which is a fixed point of the map f. Hence the map f has no points of period 8 or any other periods that precede it in the Sharkovsky order.

(c) To construct maps that have points of period 2^n but no points of period 2^{n+1}, we use the double map g that was used previously in Case II. Here we start with the map f defined in Case IIIb, which has points of period 2^2 but no points of period 2^3. The double map g: $[1,10] \to [1,10]$ is defined as follows:

$$g(x) = \begin{cases} f(x) + 6, & \text{for } 1 \leq x \leq 4 \\ x - 6, & \text{for } 7 \leq x \leq 10. \end{cases}$$

Then the map g has points of period 2^3 but no points of period 2^4 (Figure 5).

This construction can be carried out indefinitely to produce maps that have points of period 2^n but no points of period 2^{n+1}

Remark. There is still one more question to be settled. Can we construct maps that have points of period $2^n \times 3$ but no points of any period of the form $2^{n-1} \times$ odd integer? Fortunately, using the double map one can give an affirmative answer to this question. Let us first construct a map that has points of period 2×3 but no points of odd periods. Define f: $[1,3] \to [1,3]$ by letting $f(1) = 2$, $f(2) = 3$, and $f(3) = 1$ and on each interval $[n, n+1]$ we assume f to be

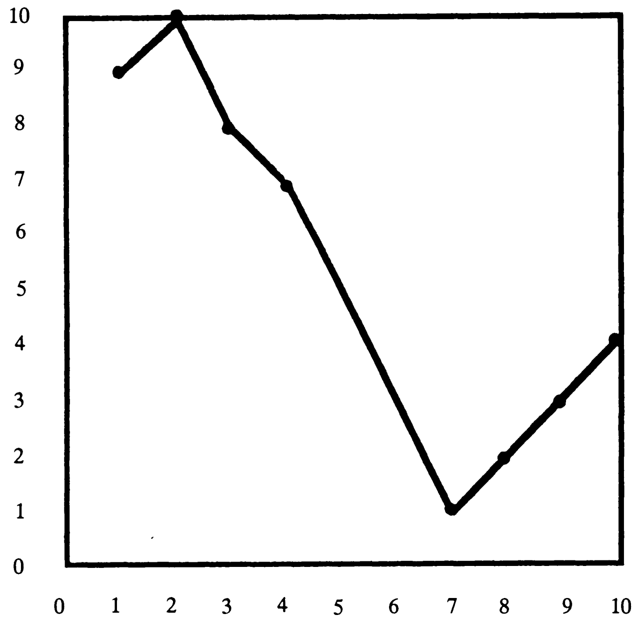

Figure 5

linear. Clearly the points 1, 2, and 3 are all of period 3. Now the double map g: $[1,7] \to [1,7]$ is defined as

$$g(x) = \begin{cases} f(x) + 4, & \text{for } 1 \le x \le 3 \\ x - 4, & \text{for } 5 \le x \le 7. \end{cases}$$

Observe that the map g has points of period 2×3 but no points of odd period. By repeating this process, one can construct continuous maps that have points of period $2^n \times 3$ but no points of period $2^{n-1} \times$ odd integer.

Addendum. After writing this note, I was informed by Dr. Hasfura of Trinity University that Delahaye **[1]** had an example of a continuous map that has points of period 2^n for all nonnegative integers n and no other periods. For the sake of completion, I include this example here.

Example. Let $I = [0,1]$ and $I_k = [1 - 1/3^k, 1 - 2/3^{k+1}]$, for all $k \ge 0$. For each k let f_k: $I_k \to I_k$ be a continuous map. Define a continuous map f: $I \to I$ by letting $f(1) = 1$, $f(x) = f_k(x)$ if $x \in I_k$ and by linearity elsewhere. Now for each nonnegative integer k choose f_k such that it has points of period 2^k but no points of period 2^{k+1}. Then f has points of periods 2^n for all nonnegative integers n but no points of other periods.

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REFERENCES

1. Delahaye, J. P., Fonctions admittant des Cycles d'ordre n'importe quelle puissance de 2 et aucun autre cycle, *C.R. Acad. Sci. Paris*, Sér. A–B, 291 (1980) A 323–325.
2. Devaney, R., *An introduction to chaotic dynamical systems*, Second Edition, Addison-Wesley, Menlo Park, California, 1989.
3. Elaydi, S., *Chaotic dynamical systems with fractals*, Preprint.
4. Gulick, D., *Encounters with chaos*, McGraw-Hill, New York, 1992.
5. Sharkovsky, A. N. Co-existence of Cycles of a continuous mapping of a line into itself. *Ukranian Math. Z.* 16 (1964), 61–71.
6. Štefan, P., A Theorem of Sharkovsky on the existence of periodic orbits of continuous endomorphisms of the real line, *Comm. Math. Phys.* 54 (1977), 237–248.

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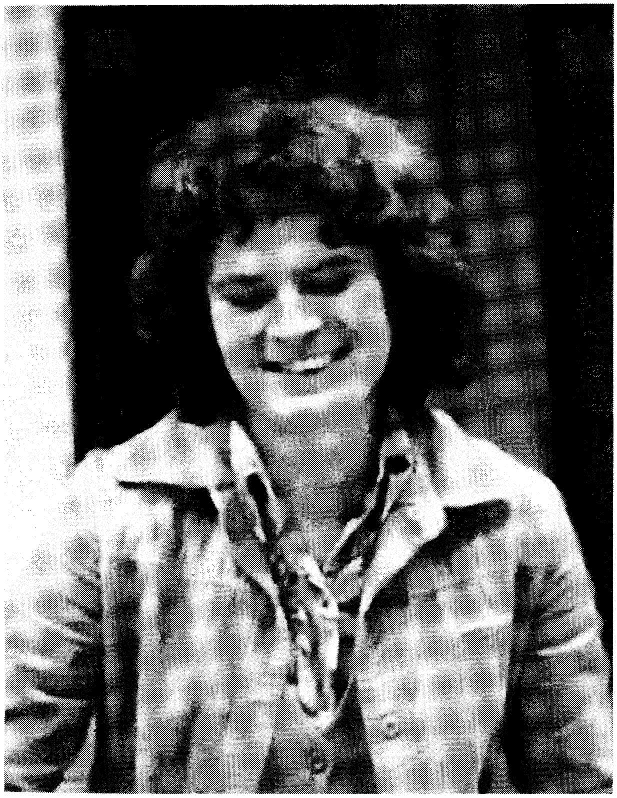
PICTURE PUZZLE

(from the collection of Paul Halmos)



. . . and a mathematician who knows physics.

(see page 440)