Devaney's chaos revisited*

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Abstract

Let X be a metric space, and let $f: X \to X$ be a continuous transformation. In this note, a concept *indecomposability* of f is introduced. We show that transitivity implies indecomposability and that Devaney's chaos is equivalent to indecomposability together with density of periodic points. Moreover, we point out that the indecomposability and the periodic-points density are independent of each other even for interval maps (i.e., neither implies the other).

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1 Introduction

Throughout this note, we let $f: X \to X$ be a continuous transformation of a metric space X. Devaney [5] called it to be *chaotic* if it satisfies the following three conditions:

(i) f is transitive.

(ii) the periodic points of f are dense in X.

(iii) f has sensitive dependence on initial conditions.

As the concepts entropy and Li-Yorke's chaos, Devaney's chaos is an important tool to discover the complexity of the dynamical system (X, f). These three concepts have intrinsic relations each other [1, 4, 7].

It is well known that in Devaney's chaos, conditions (i), (ii) and (iii) are not independent of each other; see, for examples, Banks et al. [3], Assaf IV and Gadbois [2], Vellekoop and Berglund [9], Crannell [6], and Touhey [8].

In this note, we further study Devaney's chaos by introducing the concept—*indecomposability*. Two f-invariant closed subsets A, B are *independent* if they have no common interior points; that is, $\mathrm{Int}(A) \cap \mathrm{Int}(B) = \varnothing$. Now (X, f) is called *indecomposable* if any two f-invariant closed subsets having nonempty interiors are not independent.

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Since transitivity is equivalent to the fact that the only f-invariant closed subset of X having nonempty interior is X itself, transitivity implies indecomposability. The converse however is not true even in the case of $X = [0, 1]$, as shown by Example 3.1 below.

Our main result proved in this note can be stated as follows:

Main Theorem. *For the topological dynamical system (X, f), the following two statements are equivalent to each other:*

(1) *f is Devaney chaotic.*

(2) *f is indecomposable and the periodic points are dense in X.*

We will prove this in Section 2. An interesting point is that the indecomposability and the periodic-points density are independent of each other even for interval maps (i.e., neither implies the other), as shown by Example 3.2 below.

2 Several equivalent definitions for Devaney's chaos

Let (X, f) be a topological dynamical system as in Section 1. To avoid the trivial case, we assume that X has at least infinitely many elements.

We denote respectively the recurrent points set, orbit and ω-limit points set of f by

$$R(f) = \{x \in X \mid \exists n_k \uparrow \infty \text{ s.t. } f^{n_k}(x) \to x\},$$
$$\mathrm{Orb}_f(x) = \{f^n(x) | n \geq 0\},$$
$$\omega(x) = \bigcap_{n \geq 0} \overline{\{f^k(x) | k \geq n\}}.$$

Given a subset $A \subset X$, we denote the interior of A by $\mathrm{Int}(A)$ and the closure of A by $\overline{A}$ in X. A is invariant if $f(A) \subset A$. A subset S of X is residual if it contains a dense G_δ set. A Baire space is a topological space such that every nonempty open subset is of second category.

Recall that f is transitive if for any two nonempty open subsets U and V in X there exists $n \in \mathbb{Z}_+$ such that $f^n(U) \cap V \neq \varnothing$. A point $x \in X$ is called a transitive point if $\overline{\mathrm{Orb}_f(x)} = X$. By Tr_f we mean the set of all transitive points of f. It is well known that Tr_f is a dense G_δ set if X is a Baire separable metric space.

Definition 2.1. Let $f\colon X \to X$ be a continuous transformation on the metric space X. f is said to be

(i) *strongly indecomposable* if for any sequence of f-invariant closed subsets $\{A_n\}_{n=1}^\infty$ of X with $\mathrm{Int}(A_n) \neq \varnothing$, $\mathrm{Int}(\bigcap_{n=1}^\infty A_n) \neq \varnothing$;

(ii) *indecomposable* if for any two f-invariant closed subsets $A, B \subset X$ with $\mathrm{Int}(A) \neq \varnothing$ and $\mathrm{Int}(B) \neq \varnothing$, $\mathrm{Int}(A \cap B) \neq \varnothing$;

(iii) *weakly indecomposable* if there exists a residual subset $S \subset X$ such that for any two points $x, y \in S$, $\omega(x) = \omega(y) \neq \varnothing$.

It is easily seen that the following implication relations hold:

$$transitivity \Rightarrow strongly\ \ indecomposability \Rightarrow indecomposability.$$

We will show that indecomposability implies weakly indecomposability provided that X is a compact space (see Theorem 3.1 below). And we will give examples in Section 3 to show all the converses are not true.

Lemma 2.2. *Let $f \colon X \to X$ be a continuous transformation on the metric space X such that $X = \overline{R(f)}$. Then the following conditions are equivalent:*

(1) *f is transitive.*

(2) *f is strongly indecomposable.*

(3) *f is indecomposable.*

Proof. Obviously, (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Now we prove (3) $\Rightarrow$ (1). It suffices to show that for any closed invariant subset A of X with nonempty interior, we have $A = X$, under the condition $X = \overline{R(f)}$. In fact, as f is indecomposable, for any nonempty open set $V \subset X$, we have $\overline{\bigcup_{n\geq 0} f^n(V)} \cap A$ has nonempty interior. Then there exist nonempty open set $V_1 \subset V$ and $n \in \mathbb{Z}_+$ such that $f^n(V_1) \subset \mathrm{Int}(A)$. Since the recurrent points of f are dense in V_1 and A is an invariant closed set, we have $V_1 \subset A$. By the arbitrariness of V, we get $X \subset A$. Thus, f is transitive.

This proves Lemma 2.2. □

Lemma 2.3. *Let $f \colon X \to X$ be a continuous transformation on a Baire separable metric space X such that $X = \overline{R(f)}$. Then the following conditions are equivalent:*

(1) *f is transitive.*

(2) *f is strongly indecomposable.*

(3) *f is indecomposable.*

(4) *f is weakly indecomposable.*

Proof. According to Lemma 2.2, we need only prove (1) $\Leftrightarrow$ (4). Since X is a Baire separable metric space, Tr_f is a dense G_δ set. For any two points $x, y \in \mathrm{Tr}_f$, $\omega(x) = \omega(y) = X$. Thus (1) $\Rightarrow$ (4) holds. Conversely, assume f is weakly indecomposable. Let S be the residual set such that for any two points $x, y \in S$, $\omega(x) = \omega(y)$. since $R(f)$ is a dense G_δ set, $S \cap R(f)$ is residual. For any points $x, y \in S \cap R(f)$, y is recurrent and $y \in \omega(y) = \omega(x)$. As ω-limit set of x is closed, we have $\omega(x) = X$. Thus f is transitive.

This proves Lemma 2.3. □

From the statements of Lemmas 2.2 and 2.3, we easily get the following two results.

Theorem 2.4. *Let $f\colon X\to X$ be a continuous transformation on the metric space X. Then the following conditions are equivalent:*

1. *f is Devaney chaotic.*
2. *f is transitive and has a dense set of periodic points.*
3. *f is strongly indecomposable and has a dense set of periodic points.*
4. *f is indecomposable and has a dense set of periodic points.*

Theorem 2.5. *Let $f\colon X\to X$ be a continuous transformation on a Baire separable metric space X. Then the following conditions are equivalent:*

1. *f is Devaney chaotic.*
2. *f is weakly indecomposable and has a dense set of periodic points.*

Thus Theorem 2.4 implies our Main Theorem stated in Section 1.

3 Indecomposability and periodic-points density

Let $f\colon X\to X$ be a continuous transformation on the metric space X. From now on, we let $U^*=\overline{\bigcup_{n\geq 0} f^n(U)}$ for any $U\subset X$, which is an invariant closed set of f. Firstly we show that indecomposability implies weakly indecomposability provided that X is compact.

Theorem 3.1. *Suppose X is compact. If f is indecomposable, then f is weakly indecomposable.*

Proof. Let $\mathbb{B}=\{U_i\}_{i=1}^{\infty}$ be a topology basis of X. As f is indecomposable, for any $k\in\mathbb{Z}_+$, $\bigcap_{i=0}^{k} U_i^*\neq\varnothing$. It follows from the compactness of X that $E=\bigcap_{i=0}^{\infty} U_i^*$ is nonempty, closed and invariant. Let $\mathbb{B}_E=\{U\in\mathbb{B}|U\cap E\neq\varnothing\}$. Then for any $U\in\mathbb{B}_E$, $\bigcup_{n=0}^{\infty} f^{-n}(U)$ is open and dense in X. Thus $V_1=\bigcap_{U\in\mathbb{B}_E}\bigcup_{n=0}^{\infty} f^{-n}(U)$ is a dense G_δ set of X. Taking any $x\in V_1$, we have $\overline{\mathrm{Orb}_f(x)}\supset E$.

For any positive integer k, let $L_k=\mathrm{Int}(\bigcap_{i=1}^{k} U_i^*)$ and $\Delta_k=\bigcup_{i=0}^{\infty} f^{-i}(L_k)$. As f is indecomposable, we have $L_k\neq\varnothing$. Thus Δ_k is open and dense in X, and $\omega(x)\subset\bigcap_{i=1}^{k} U_i^*$ for any $x\in\Delta_k$. By Baire's theorem, it follows that $V_2=\bigcap_{k=1}^{\infty}\Delta_k$ is a dense G_δ set. Therefore for any point $x\in V_2$, $\omega(x)\subset\bigcap_{i=1}^{\infty} U_i^*=E$.

We know that $V_1\cap V_2$ is a dense G_δ set from its construction. We also have $\omega(x)\subset E\subset\overline{\mathrm{Orb}_f(x)}$ for any $x\in V_1\cap V_2$. There are the following two cases.

Case 1, $\mathrm{Int}(E)\neq\varnothing$. Take $x\in\mathrm{Int}(E)\cap V_1\cap V_2$ and $y\in V_1\cap V_2$. If x is an isolate point, then $f^n(y)=x$ for some n and $\omega(x)=\omega(y)$. Otherwise x must be a recurrent point of f, then $E\supset\omega(y)\supset\omega(x)=E$, which implies $\omega(x)=\omega(y)$. Thus f is weakly indecomposable.

Case 2, $\mathrm{Int}(E)=\varnothing$. Let $E_0=\{e\in E|e \text{ is isolate in } E\}$. That is, E_0 is the set of all points e in E with $B_\varepsilon(e)\cap E=e$ for some $\varepsilon>0$, where $B_\varepsilon(e)$ stands for open ball of center e and radius

ε. E_0 must be countable. We claim that $f^{-n}(e)$ has empty interior for each $n > 0$ and $e \in E$. If not, let n be the smallest positive integer such that $f^{-n}(e)$ has nonempty interior. When $f^{-n}(e)$ is a singleton, $f^{-n}(e) \in U_i^*$ for each i since $\overline{\bigcup_{i \geq -n} f^i(e)}$ is an invariant closed subset with its interior being $f^{-n}(e)$ and f is indecomposable. Therefore $f^{-n}(e) \in E$ which contradicts $\mathrm{Int}(E) = \varnothing$. When $f^{-n}(e)$ is not a singleton, taking two disjoint nonempty open subset $\alpha, \beta \subset f^{-n}(e)$, we have $\mathrm{Int}(\alpha^* \cap \beta^*) = \varnothing$ which contradicts to the indecomposability of f.

Since $f^{-n}(e)$ is a closed set with empty interior for $n > 0$ and $e \in E$, we have that $V_3 = \bigcup_{e \in E_0} \bigcup_{n=0}^{\infty} f^{-n}(e)$ is of first category and so $V_1 \cap V_2 - V_3$ is a residual set. Take a point $x \in V_1 \cap V_2 - V_3$, if $Orb_f(x) \cap E = \varnothing$, then $E \subset \overline{Orb_f(x)} - Orb_f(x) \subset \omega(x) \subset E$. Otherwise there exists the smallest nonnegative integer n such that $f^n(x) \in E$. Then we have

$$\omega(x) \subset E \subset \overline{Orb_f(x)} - \{x,\ f(x), \cdots,\ f^{n-1}(x)\} = \overline{Orb_{f^n}(x)}.$$

And $f^n(x)$ must be a cluster point of E because $x \notin V_3$. Thus $f^n(x)$ is a recurrent point and

$$\omega(x) \subset E \subset \overline{Orb_{f^n}(x)} = \omega(f^n(x)) = \omega(x).$$

Therefore, in both cases we always have $\omega(x) = E$. This means that f is weakly indecomposable.

This completes the proof of Theorem 3.1. □

Secondly, we show that strongly indecomposability is nearly transitivity.

Theorem 3.2. *If X is a Baire separable metric space with no isolate point, $f : X \to X$ is strongly indecomposable, then there exist an invariant closed set E with nonempty interior such that $f|_E$ is transitive.*

Proof. Let $\mathbb{B} = \{U_i\}_{i=1}^{\infty}$ be a topology basis of X and $E = \bigcap_{i=1}^{\infty} U_i^*$. Then $\mathrm{Int}(E) \neq \varnothing$ by the strongly indecomposability of f. Let $\mathbb{B}_E = \{U \in \mathbb{B} | U \cap E \neq \varnothing\}$. Then for any $U \in \mathbb{B}_E$, $\bigcup_{n=0}^{\infty} f^{-n}(U)$ is open and dense in X, which implies that $\bigcap_{U \in \mathbb{B}_E} \bigcup_{n=0}^{\infty} f^{-n}(U)$ is a dense G_δ set of X. Every point x in $\mathrm{Int}(E) \cap \bigcap_{U \in \mathbb{B}_E} \bigcup_{n=0}^{\infty} f^{-n}(U)$ is a transitive point of $f|_E$. Therefore, $f|_E$ is transitive since X has no isolate point. □

Finally, we consider one-dimensional system (I, f), where I is an interval and f is a continuous map of I. We need a lemma.

Lemma 3.3 ([9]). *Suppose $f : I \to I$ is an interval map. If $J \subset I$ is a subinterval containing no periodic point and $z, f^m(z), f^n(z) \in J$ with $0 < m < n$, then either $z < f^m(z) < f^n(z)$ or $z > f^m(z) > f^n(z)$.*

Theorem 3.4. *An interval map $f : I \to I$ is strongly indecomposable, then there exist a positive integer n and disjoint closed non degenerate subintervals $J_0, J_1, \ldots, J_{n-1}, J_n = J_0$ such that $f(J_i) = J_{i+1},\ i = 0, \ldots, n-1$ and f is Devaney chaotic on $\bigcup\limits_{i=0}^{n-1} J_i$. Furthermore, f^n is Devaney chaotic on $J_i,\ i = 0, \ldots, n-1$.*

Proof. Assume that f is strongly indecomposable. By Theorem 3.2, there exists a closed subset E which contains a non degenerate interval J such that $f|_E$ is transitive. Let (a,b) be any non degenerate subinterval of J. Suppose (a,b) contains no periodic point. By the transitivity of $f|_E$, there exist a transitive point $x \in (a,b)$ and $0 < p < q$ such that $a < f^q(x) < x < f^p(x) < b$, which contradicts with Lemma 3.3. Thus the periodic points are dense in J. Since $f|_E$ is transitive, the periodic points are dense in E, $f|_E$ is Devaney chaotic.

Let J_0 be the longest subinterval of E. It must be closed because E is closed. Since $f|_E$ is transitive, there exists the smallest positive integer n such that $f^n(J_0) \cap J_0 \neq \varnothing$. We have $f^n(J_0) \subset J_0$ as J_0 is the longest subinterval. The transitivity of $f|_E$ ensures $f^n(J_0) = J_0$. Let $J_i = f^i(J_0),\ i = 0, \ldots, n-1$. We claim that $J_i,\ i = 0, \ldots, n-1$ are disjoint. If not, there exist integers $0 \le l < m \le n-1$ such that $J_l \cap J_m \neq \varnothing$, which follows $J_0 \cap J_{m-l} \supset f^{n-l}(J_l \cap J_m) \neq \varnothing$. A contradiction.

Since J_0 is closed and $f^{n-i}(J_i) = J_0$, J_i is closed for $i = 0, \ldots, n-1$. It follows that $\bigcup\limits_{i=0}^{n-1} J_i$ is invariant and closed. So $E = \bigcup\limits_{i=0}^{n-1} J_i$. It's not difficult to check that $f^n|_{J_i}$ is transitive and chaos in the sense of Devaney. □

At the end of this note, we give two examples to illustrate that

$$\text{indecomposability} \not\Rightarrow \text{strongly indecompossability} \not\Rightarrow \text{transitivity}.$$

Example 3.1. Let $I = [0,1]$ and f be defined as

$$f(x) = \begin{cases} -2x+1, & x \in [0, \frac{1}{6}], \\ 2x+1/3, & x \in [\frac{1}{6}, \frac{1}{3}], \\ -3x+2, & x \in [\frac{1}{3}, \frac{2}{3}], \\ x-2/3, & x \in [\frac{2}{3}, 1]. \end{cases}$$

See Figure 3.1. Then f is strongly indecomposable but not transitive.

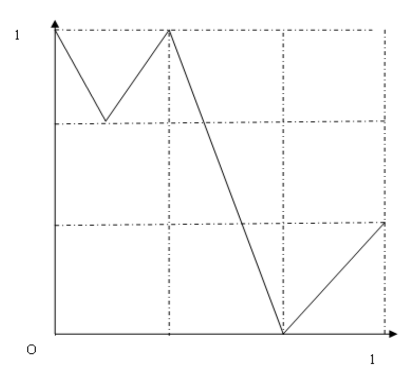

Figure 3.1: The profile of f in Example 3.1

Proof. We show that f is strongly indecomposable. On interval $[0, \frac{1}{3}]$, f^2 can be expressed as

$$f^2(x) = \begin{cases} -2x + \frac{1}{3}, & x \in [0, \frac{1}{6}], \\ 2x - \frac{1}{3}, & x \in [\frac{1}{6}, \frac{1}{3}]. \end{cases}$$

It is clear that $f^2|_{[0,\frac{1}{3}]}$ is mixing. For any non degenerate subinterval $J \subset [0, 1]$, there exists an integer $n \geq 0$ such that $f^n(J) \cap (0, \frac{1}{3}) \neq \varnothing$. Since $f^n(J)$ is non degenerate, $\overline{\bigcup_{n\geq 0} f^n(J)} \supset [0, \frac{1}{3}]$, f is strongly indecomposable on $[0, 1]$. On the other hand, for $x \in (\frac{3}{9}, \frac{4}{9})$, $f^n(x)$ never comes back into $(\frac{1}{3}, \frac{2}{3})$ any more for $n > 0$. Thus f is not transitive. □

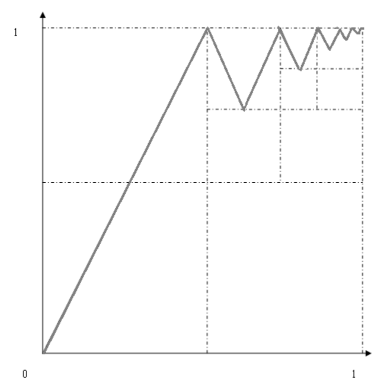

Figure 3.2: The profile of f on [0,1] in Example 3.2.

Example 3.2. Let $I = [0, 1]$ and $f : I \to I$ be defined as

$$f(0) = 0; f(1) = 1;$$
$$f(1 - \frac{1}{2^n}) = 1, \quad n = 1, 2, \dots,$$
$$f(1 - \frac{3}{2^{n+2}}) = 1 - \frac{1}{2^{n+1}}, \quad n = 1, 2, \dots.$$

f is linear between $1 - \frac{1}{2^n}$ and $1 - \frac{3}{2^{n+2}}$, $n = 1, 2, \dots$. See Figure 3.2. Then f is indecomposable but not strongly indecomposable. Furthermore, f has only two periodic points 0 and 1.

Proof. To show that f is indecomposable, let $A, B \subset X$ be two invariant closed subsets with non degenerate intervals $I \subset A$, $J \subset B$, respectively. If I covers at least 2 critical points, then $f(I) \supset [1 - \frac{|I|}{4}, 1]$. Here $|I|$ denotes the length of the interval I. If I covers less than two critical points, we have $|f(I)| \geq |I|$ by the fact that the absolute value of slope of f is 2 everywhere except the critical points. Thus there exists the smallest positive integer n such that $f^n(I)$ covers at least two critical points. We have $f^{n+1}(I) \supset [1 - \frac{|I|}{4}, 1]$. Hence $A \supset [1 - \frac{|I|}{4}, 1]$. Similarly, we have $B \supset [1 - \frac{|J|}{4}, 1]$. Therefore, $A \cap B$ contains a non degenerate interval and f is indecomposable.

But f is not strongly indecomposable. In fact, let $J_n = [1 - \frac{1}{2^n}, 1], n = 1, 2, \dots$. Then J_n is invariant with nonempty interiors and $\bigcap_n J_n = \{1\}$ which contains no interior point.

It is easily seen that the periodic points of f are $\{0, 1\}$ and the point $\{1\}$ attracts all the points except the origin. f is far from chaos. □

Remark 3.1. For interval maps $f : I \to I$, strongly indecomposability does not imply periodic-points density on I, but it ensures Devaney's chaos on some subintervals of I.

Remark 3.2. Even for interval maps, indecomposability does not imply periodic-points density. Example 3.2 demonstrates that an indecomposable interval map can be far from chaos.

Remark 3.3. Weakly indecomposability is the weakest concept among the three ones on compact space. Such system has a topologically "large" set of points, each of which has the same ω-limit set. An indecomposable map can have very simple dynamics. For example, any constant map on a metric space X (a map which maps all points of X into a common fixed point) is indecomposable.

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