PERIODIC ORBITS OF BILLIARDS ON AN EQUILATERAL TRIANGLE

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ABSTRACT. Using elementary methods, we find, classify and count the classes of periodic orbits of a given period on an equilateral triangle. A periodic orbit is either primitive or some iterate of a primitive orbit. Every periodic orbit with odd period is some odd iterate of Fagnano's period 3. Let μ denote the Möbius function. For each $n \in \mathbb{N}$, there are exactly $\sum_{d|n} \mu(d) \left(\lfloor \frac{n/d+2}{2} \rfloor - \lfloor \frac{n/d+2}{3} \rfloor \right)$ classes of primitive orbits with period $2n$.

1. INTRODUCTION

The trajectory of a billiard ball in motion on a frictionless billiards table is completely determined by its initial position, direction and speed. When the ball strikes a bumper, we assume that the angle of incidence equals the angle of reflection. Once released, the ball continues indefinitely along its trajectory with constant speed unless it strikes a vertex, at which point it stops. If the ball returns to its initial position with its initial velocity direction, it retraces its trajectory and continues to do so repeatedly; we call such trajectories *periodic*. An *orbit* is any path of the ball. An orbit is *periodic* if the ball follows a periodic trajectory through exactly $n \geq 1$ complete retracings. If $n = 1$, the orbit is *primitive*; otherwise it is an *n-fold iterate.* If α denotes a primitive orbit, α^n denotes its n-fold iterate. The *period* of an orbit is the number of times the ball strikes a bumper as it travels along its trajectory. If α is primitive of period k, then α^n has period kn. Non-periodic orbits are either *infinite* or *singular*; in the later case the ball strikes a vertex.

In this article we give a complete solution to the following billiards problem: *Find, classify and count the classes of periodic orbits of a*

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given period on an equilateral triangle. While periodic orbits are known to exist on all non-obtuse and certain classes of obtuse triangles [4] , [7], [10], [13], existence in general remains a long-standing open problem. The first examples of periodic orbits were discovered by Fagnano in 1745. Interestingly, his orbit of period 3 on an acute triangle, known as the "Fagnano orbit," was not found as the solution of a billiards problem, but rather as the triangle of least perimeter inscribed in a given acute triangle. This problem, known as "Fagnano's problem," is solved by the orthic triangle, whose vertices are the feet of the altitudes of the given triangle (see Figure 1). The orthic triangle is a billiard orbit since its angles are bisected by the altitudes of the triangle in which it is inscribed; the proof given by Coxeter and Greitzer [1] uses exactly the "unfolding" technique we apply below. Coxeter credits this technique to H. A. Schwarz and mentions that Frank and F. V. Morley [8] extended Schwarz's treatment on triangles to odd-sided polygons. For a discussion of some interesting properties of the Fagnano orbit on any acute triangle, see [3].

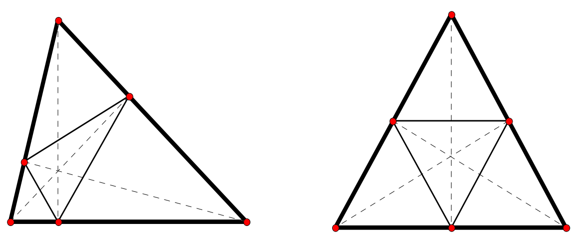

FIGURE 1. Fagnano's period 3 orbit.

Much later, in 1986, Masur [7] proved that every *rational* polygon (one whose interior angles are rational multiples of π) admits infinitely many periodic orbits with distinct periods, but he neither constructs nor classifies them. A year later Katok [5] proved that the number of periodic orbits of a given period grows subexponentially. Existence results on various polygons were compiled by Tabachnikov [12] in 1995.

This article is organized as follows: In Section 2 we introduce an equivalence relation on the set of all periodic orbits on an equilateral triangle and prove that every orbit with odd period is an odd iterate of Fagnano's orbit. In Section 3 we use techniques from analytic geometry to identify all orbits and classify them. The paper concludes with Section 4, in which we derive two counting formulas: First, we establish a bijection between classes of orbits with period $2n$ and partitions of n with 2 or 3 as parts and use it to show that there are $\mathcal{O}(n) = \lfloor \frac{n+2}{2} \rfloor - \lfloor \frac{n+2}{3} \rfloor$ classes of orbits with period $2n$ (counting iterates). Second, we show that there are $\mathcal{P}(n) = \sum_{d|n} \mu(d)\mathcal{O}(n/d)$ classes of primitive orbits with period $2n$, where μ denotes the Möbius function.

2. Orbits and Tessellations

Consider an equilateral triangle $\triangle ABC$. We begin with some key observations.

Proposition 1. *Every non-singular orbit strikes some side of $\triangle ABC$ with an angle of incidence in the range $30^\circ \leq \theta \leq 60^\circ$.*

Proof. Given a non-singular orbit α, choose a point P_1 at which α strikes $\triangle ABC$ with angle of incidence θ_1. If θ_1 lies in the desired range, set $\theta = \theta_1$. Otherwise, let α_1 be the segment of α that connects P_1 to the next strike point P_2 and label the vertices of $\triangle ABC$ so that P_1 is on side $\overline{AC}$ and P_2 is on side $\overline{BC}$. If $60^\circ < \theta_1 \leq 90^\circ$, the angle of incidence at P_2 is $\theta_2 = m\angle P_1P_2C = 120^\circ - \theta_1$ and satisfies $30^\circ \leq \theta_2 < 60^\circ$ (see Figure 2). So set $\theta = \theta_2$. If $0^\circ < \theta_1 < 30^\circ$, then $\theta_2 = m\angle P_1P_2B = \theta_1 + 60^\circ$ so that $60^\circ < \theta_2 < 90^\circ$. Let α_2 be the segment of α that connects P_2 to the next strike point P_3. Then as in the previous case, the angle of incidence at P_3 satisfies $30^\circ < \theta_3 < 60^\circ$; set $\theta = \theta_3$. $\square$

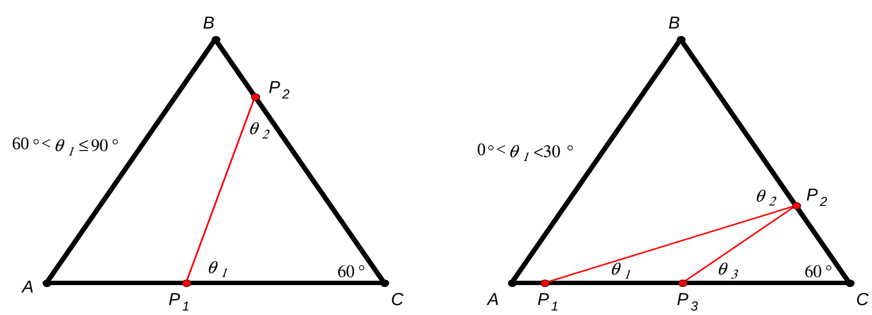

Figure 2. Incidence angles θ_2 (left) and θ_3 (right) in the range $30^\circ \leq \theta \leq 60^\circ$.

Let α be an orbit of period n on $\triangle ABC$, and choose a point P at which α strikes $\triangle ABC$ with angle of incidence in the range $30^\circ \leq \theta \leq 60^\circ$. Without loss of generality, we may assume that α begins and ends at P. If necessary, relabel the vertices of the triangle so that side $\overline{BC}$ contains P. Let $\mathcal{T}$ be a regular tessellation of the plane by equilateral triangles, each congruent to $\triangle ABC$ and positioned so that one of its families of parallel edges is horizontal. Embed $\triangle ABC$ in $\mathcal{T}$ so that its base $\overline{BC}$ is collinear with a horizontal edge of $\mathcal{T}$. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ denote the directed segments of α, labelled sequentially; then α_1 begins at P and terminates at P_1 on side s_1 of $\triangle ABC$ with angle of incidence θ_1. Let σ_1 be the reflection in the edge of $\mathcal{T}$ containing s_1. Then α_1 and $\sigma_1(\alpha_2)$ are collinear segments and $\sigma_1(\alpha)$ is a periodic orbit on $\sigma_1(\triangle ABC)$, which is the basic triangle of $\mathcal{T}$ sharing side s_1 with $\triangle ABC$. Follow $\sigma_1(\alpha_2)$ from P_1 until it strikes side s_2 of $\sigma_1(\triangle ABC)$ at P_2 with incidence angle θ_2. Let σ_2 be the reflection in the edge of $\mathcal{T}$ containing s_2; then α_1, $\sigma_1(\alpha_2)$ and $(\sigma_2\sigma_1)(\alpha_3)$ are collinear segments and $(\sigma_2\sigma_1)(\alpha)$ is a periodic orbit on $(\sigma_2\sigma_1)(\triangle ABC)$. Continuing in this manner for $n-1$ steps, let θ_n be the angle of incidence at $Q = (\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1)(P)$. Then $\theta_1, \theta_2, \ldots, \theta_n$ is a sequence of incidence angles with $30^\circ \leq \theta_n \leq 60^\circ$, and $\alpha_1, \sigma_1(\alpha_2), \ldots, (\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1)(\alpha_n)$ is a sequence of collinear segments whose union is the directed segment from P to Q. Using the notation in [6], let $\underline{PQ}$ denote the directed segment from P to Q. Then $\underline{PQ}$ has the same length as α and enters and exits the triangle $(\sigma_i\cdots\sigma_1)(\triangle ABC)$ with angles of incidence θ_i and θ_{i+1}. We refer to $\underline{PQ}$ as an *unfolding* of α and to θ_n as its *representation angle*.

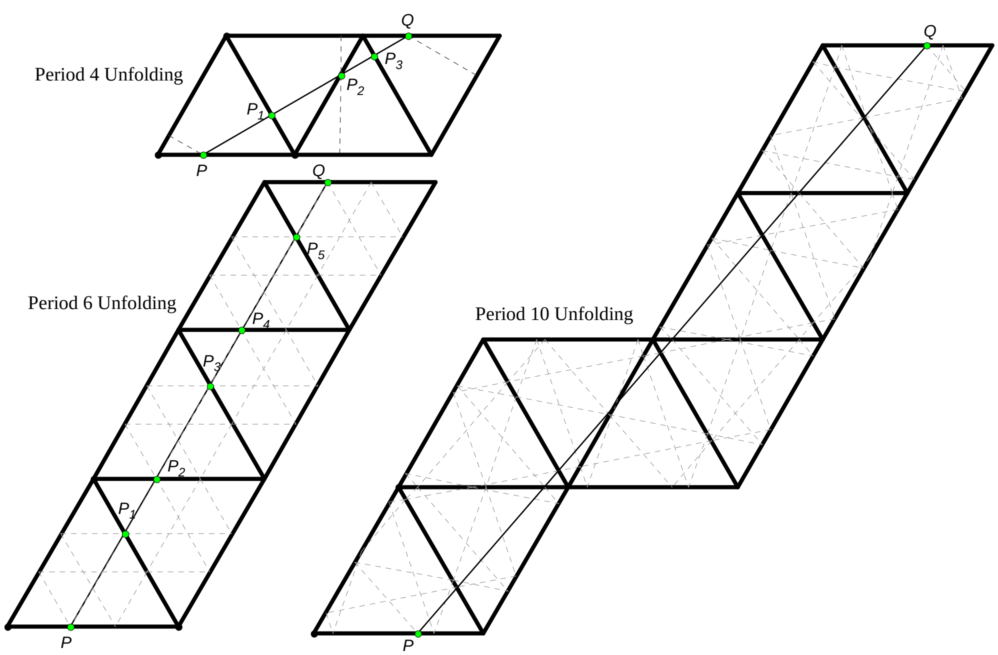

FIGURE 3. Unfolded orbits of period 4, 6, and 10.

Proposition 2. *A periodic orbit strikes the sides of* $\triangle ABC$ *with at most three incidence angles, exactly one of which lies in the range* $30° \leq \theta \leq 60°$. *In fact, exactly one of the following holds:*

(1) *All incidence angles measure* $60°$.
(2) *There are exactly two distinct incidence angles measuring* $30°$ *and* $90°$.
(3) *There are exactly three distinct incidence angles* ϕ, θ *and* ψ *such that* $0° < \phi < 30° < \theta < 60° < \psi < 90°$.

Proof. Let α be a periodic orbit and let $\underline{PQ}$ be an unfolding. By construction, $\underline{PQ}$ crosses each horizontal edge of $\mathcal{T}$ with angle of incidence in the range $30° \leq \theta \leq 60°$. Consequently, $\underline{PQ}$ crosses a left-leaning edge of $\mathcal{T}$ with angle of incidence $\phi = 120° - \theta$ and crosses a right-leaning edge of $\mathcal{T}$ with angle of incidence $\psi = 60° - \theta$ (see Figure 4). In particular, if $\theta = 60°$, $\underline{PQ}$ crosses only left-leaning and horizontal edges, and all incidence angles are equal. In this case, α is either the Fagnano orbit, a primitive orbit of period 6 or some iterate of these. If $\theta = 30°$, then $\phi = 90°$ and $\psi = 30°$, and α is either primitive of period 4 or some iterate thereof (see Figure 3). When $30° < \theta < 60°$, and clearly, $0° < \phi < 30°$ and $60° < \psi < 90°$. $\square$

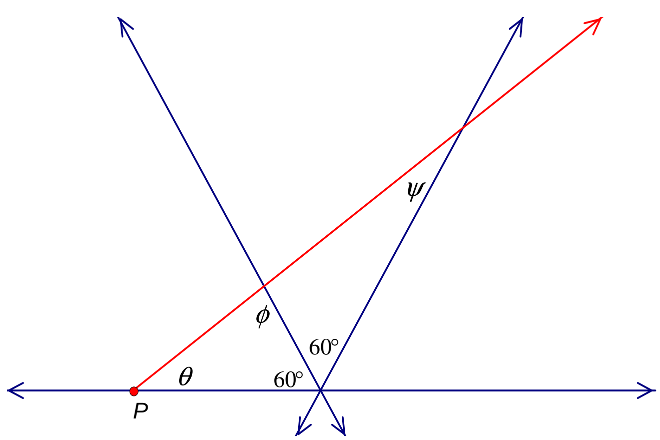

FIGURE 4. Incidence angles θ, ϕ and ψ.

Corollary 1. *Any two unfoldings of a non-singular orbit are parallel.*

Our next result plays a pivotal role in the classification of orbits.

Theorem 1. *If an unfolding of a periodic orbit* α *terminates on a horizontal edge of* $\mathcal{T}$, *then* α *has even period.*

Proof. Let $\underline{PQ}$ be an unfolding of α. Then both P and Q lie on horizontal edges of $\mathcal{T}$, and the basic triangles of $\mathcal{T}$ cut by $\underline{PQ}$ pair off and form a polygon of rhombic tiles containing $\underline{PQ}$ (see Figure 5). As the path $\underline{PQ}$ traverses this polygon, it enters each rhombic tile through an edge, crosses a diagonal of that tile (collinear with a left-leaning edge of $\mathcal{T}$), and exits through another edge. Since each exit edge of one tile is the entrance edge of the next and the edge containing P is identified with the edge containing Q, the number of distinct edges of $\mathcal{T}$ cut by $\underline{PQ}$ is twice the number of rhombic tiles. Thus α has even period. $\square$

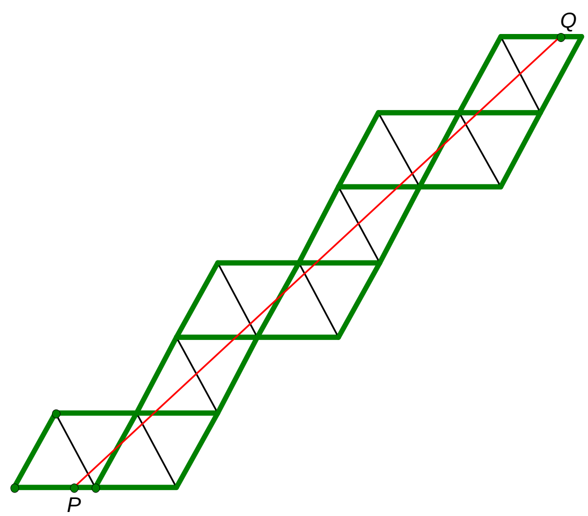

FIGURE 5. A typical rhombic tiling.

Let γ denote the Fagnano orbit.

Theorem 2. *If α is a periodic orbit and $\alpha \neq \gamma^{2k-1}$ for all $k \geq 1$, then every unfolding of α terminates on a horizontal edge of $\mathcal{T}$.*

Proof. We prove the contrapositive. Suppose there is an unfolding $\underline{PQ}$ of α that does not terminate on a horizontal edge of $\mathcal{T}$. Let θ be the angle of incidence at Q; then θ is also the angle of incidence at P and $\theta \in \{30°, 60°\}$ by the proof of Proposition 2. But if $\theta = 30°$, then α is some iterate of the period 4 orbit whose unfoldings terminate on a horizontal of $\mathcal{T}$ (see Figure 3). So $\theta = 60°$. But α is neither an iterate of the period 6 orbit nor an even iterate of γ since their unfoldings also terminate on a horizontal edge of $\mathcal{T}$ (see Figure 3). It follows that $\alpha = \gamma^{2k-1}$ for some $k \geq 1$. $\square$

Combining the contrapositives of Theorems 1 and 2 we obtain the following characterization:

Corollary 2. *If α is an orbit with odd period, then $\alpha = \gamma^{2k-1}$ for some $k \geq 1$, in which case the period is $6k-3$.*

Let α be an orbit with even period and let $\underline{PQ}$ be an unfolding. Let G be the group generated by all reflections in the edges of $\mathcal{T}$. Since the action of G on $\overline{BC}$ generates a regular tessellation $\mathcal{H}$ of the plane by hexagons, α terminates on some horizontal edge of $\mathcal{H}$. Let $f = \sigma_n\sigma_{n-1}\cdots\sigma_1$ denote the composition of reflections in the edges of $\mathcal{T}$ that (1) maps P to Q and (2) maps the hexagon whose base $\overline{BC}$ contains P to the hexagon whose base $\overline{B'C'}$ contains Q. Then n is even and f is either a translation by vector $\overrightarrow{PQ}$ or a rotation of 120° or 240°. But $\overline{BC} \| \overline{B'C'}$ so f is a translation and the position of Q on $\overline{B'C'}$ is exactly the same as the position of P on $\overline{BC}$.

On the other hand, periodic orbits represented by horizontal translations of an unfolding $\underline{PQ}$ are generically distinct, but have the same length and incidence angles (up to permutation) as α. Hence it is natural to think of them as equivalent.

Definition 1. *Periodic orbits α and β are* <u>*equivalent*</u> *if there exist respective unfoldings $\underline{PQ}$ and $\underline{RS}$ and a horizontal translation τ such that $\underline{RS} = \tau\left(\underline{PQ}\right)$. The symbol $[\alpha]$ denotes the equivalence class of α. The* <u>*period*</u> *of a class $[\alpha]$ is the period of its representatives; an* <u>*even*</u> *class has even period, otherwise it is* <u>*odd*</u>*.*

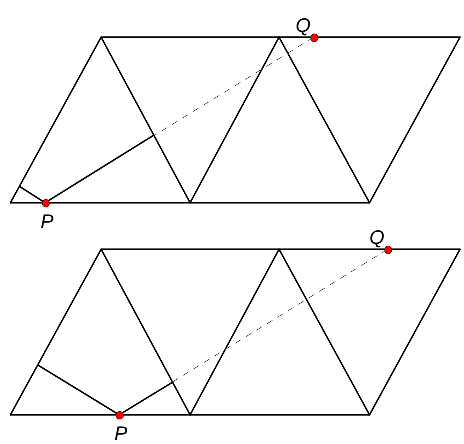

FIGURE 6. Unfoldings of equivalent period 4 orbits

Consider an unfolding $\underline{PQ}$ of a periodic orbit α. If $[\alpha]$ is even, let R be a point on $\overline{BC}$ and let τ is the translation from P to R. We say that the point R is *singular for* $[\alpha]$ if $\tau(\underline{PQ})$ contains a vertex of $\mathcal{T}$; then $\tau(\underline{PQ})$ is an unfolding of a periodic orbit whenever R is non-singular for $[\alpha]$. Furthermore, α strikes $\underline{BC}$ at finitely many points and at most finitely many points on $\overline{BC}$ are singular for $[\alpha]$. Therefore $[\alpha]$ has cardinality c (the cardinality of an interval). On the other hand, if $[\alpha]$ is odd, then $\alpha = \gamma^{2k-1}$ for some $k \geq 1$ by Corollary 2, and $\underline{PQ}$ is collinear with the midline of adjacent parallels of $\mathcal{T}$. But translations of $\underline{PQ}$ off of these midlines are not unfoldings of periodic orbits so $\left[\gamma^{2k-1}\right]$ is a singleton class. We have proved:

Proposition 3. *The cardinality of a class is determined by its parity; in fact, α has odd period if and only if $[\alpha]$ is a singleton class.*

Proposition 3 completely classifies orbits with odd period. The remainder of this article considers orbits with even period. Our strategy is to represent the classes of all such orbits as lattice points in some "fundamental region," which we now define. Since at most finitely many points in $\overline{BC}$ are singular for a given class, there is a point O on $\overline{BC}$ other than the midpoint with the following property: Given even class $[\alpha]$, there is a point S such that $\underline{OS}$ is an unfolding of some representative. Note that if $\underline{PQ}$ is an unfolding of α, then $\underline{OS}$ is the horizontal translation of $\underline{PQ}$ by $\overrightarrow{PO}$ and S is uniquely determined by α. Hence we refer to $\underline{OS}$ as the *fundamental unfolding of* $[\alpha]$. The *fundamental region at O*, denoted by Γ_O, is the polar region $30^\circ \leq \theta \leq 60^\circ$ centered at O; the points S given by fundamental unfoldings $\underline{OS}$ are called *lattice points* of Γ_O.

Since O is not the midpoint of $\overline{BC}$, odd iterates of Fagnano's orbit γ have no fundamental unfoldings. On the other hand, fundamental unfoldings of γ^{2n} represent some n-fold iterate of a primitive period 6 orbit. Nevertheless, with the notable exception of $[\gamma^2]$, "primitivity" is a property common to all orbits of the same class (see Figure 7). Indeed, the fundamental unfolding of $[\gamma^2]$ represents a primitive. So define a *primitive class* to be either $[\gamma^2]$ or a class of primitives.

To complete the classification, we must determine exactly which directed segments in Γ_O with initial point O represent orbits with even period. We address this question in the next section.

3. Orbits and Rhombic Coordinates

In this section we introduce the analytical structure we need to complete the classification and to count the distinct classes of orbits of a

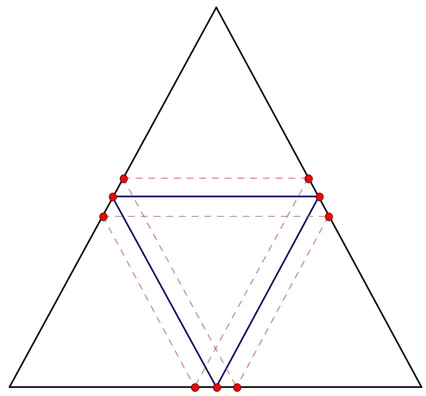

FIGURE 7. The Fagnano orbit and an equivalent period 6 orbit (dotted).

given even period. Expressing a fundamental unfolding $\underline{OS}$ as a vector $\overrightarrow{OS}$ allows us to exploit the natural rhombic coordinate system given by $\mathcal{T}$. Let O be the origin and take the x-axis to be the horizontal line containing it. Take the y-axis to be the line through O with inclination 60° and let BC be the unit of length (see Figure 8). Then in rhombic coordinates

$$\Gamma_O = \{(x, y) \mid 0 \le x \le y\}.$$

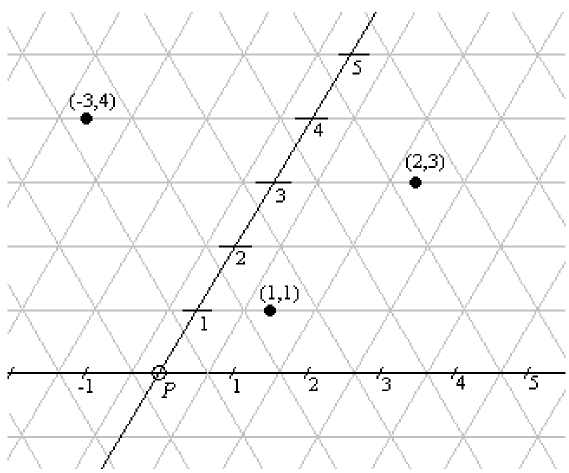

FIGURE 8. Rhombic coordinates.

Since the period of $[\alpha]$ is twice the number of rhombic tiles cut by its fundamental unfolding $\underline{OS}$, and the rhombic coordinates of S count these rhombic tiles, we can strengthen Theorem 1:

Corollary 3. *Let $\underline{OS}$ be the fundamental unfolding of $[\alpha]$. If $S = (x, y)$, then the period of α is $2(x + y)$.*

Points in the integer sublattice $\mathcal{L}$ of points on the horizontals of $\mathcal{H}$ that are images of O under the action of G have a simple characterization. Let H be the hexagon of $\mathcal{H}$ with base $\overline{BC}$; the six hexagons adjacent to H are its images $\tau_2^b \tau_1^a(H)$, $(a, b) \in \{\pm(1, 0), \pm(1, -1), \pm(2, -1)\}$. Inductively, if H' is any hexagon of $\mathcal{H}$, then $H' = \tau_2^b \tau_1^a(H)$ for some $a, b \in \mathbb{Z}$. Note that $a(1, 1)+b(0, 3)$ defines the translation $\tau_2^b \tau_1^a$. Hence $\mathcal{L}$ is generated by the vectors $(1, 1)$ and $(0, 3)$ and it follows that $(x, y) \in \mathcal{L}$ if and only if $x \equiv y \pmod 3$.

Recall that if $\underline{OS}$ is an unfolding, then S lies on a horizontal of $\mathcal{H}$. Hence $\underline{OS}$ is an unfolding if and only if $S \in \mathcal{L} \cap \Gamma_O - O$. This proves:

Theorem 3. *There is the correspondence*

$$\{[\alpha] | [\alpha] \textit{ has period } 2n\} \leftrightarrow \{(x, y) \in \mathbb{Z}^2 \cap \Gamma_O | x \equiv y (mod\ 3), x + y = n\}.$$

Taken together, Proposition 3, Corollary 3 and Theorem 3 classify all perioidic orbits on an equiateral triangle.

Theorem 4. *(Classification) Let α be a periodic orbit on an equilateral triangle.*

(1) *If α has period $2n$, then $[\alpha]$ has cardinality c and contains a unique representative whose unfolding $\underline{OS}$ with $S = (x, y)$ satisfies $0 \le x \le y$, $x \equiv y \pmod 3$ and $x + y = n$.*
(2) *Otherwise, $\alpha = \gamma^{2k-1}$ for some $k \ge 1$, in which case its period is $6k + 3$.*

In view of Theorem 3, we may count classes of orbits of a given period $2n$ by counting integer pairs (x, y) such that $0 \le x \le y$, $x \equiv y \pmod 3$ and $x+y = n$. This is the objective of the next and concluding section.

4. Orbits and Partitions of n

We will often refer to an ordered pair (x, y) as an "orbit" when we mean the even class of orbits to which it corresponds. Two questions arise: (1) Is there an orbit with period $2n$ for each $n \in \mathbb{N}$? (2) If so, exactly how many distinct classes of orbits with period $2n$ are there?

If we admit iterates, question (1) has an easy answer. Clearly there are no period 2 orbits since no two sides of $\triangle ABC$ are parallel. For each $n > 1$, the orbit

$$\alpha = \begin{cases} (\frac{n}{2}, \frac{n}{2}), & n \text{ even} \\ (\frac{n-1}{2} - 1, \frac{n-1}{2} + 2), & n \text{ odd.} \end{cases}$$

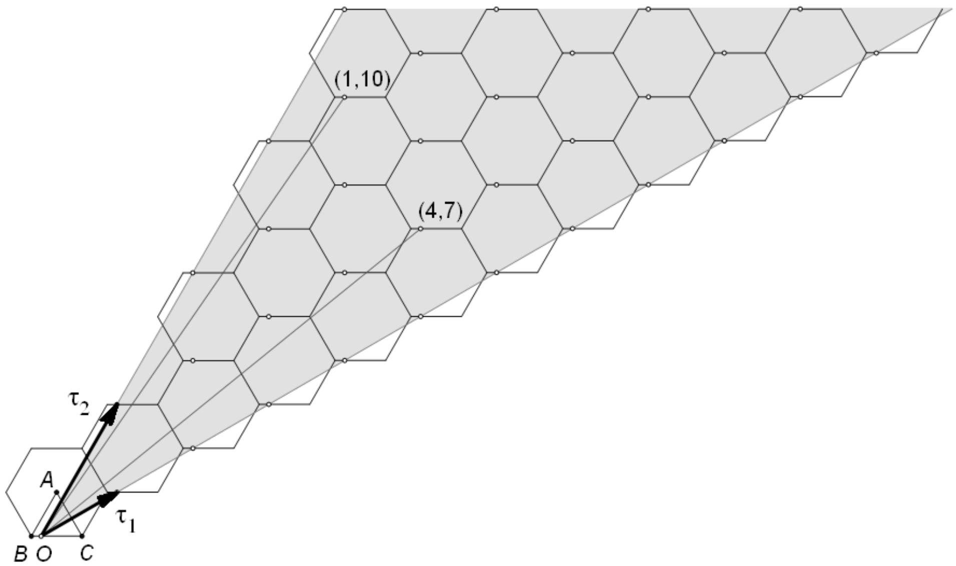

FIGURE 9. Translated images of O in Γ_O and unfoldings of period 22 orbits.

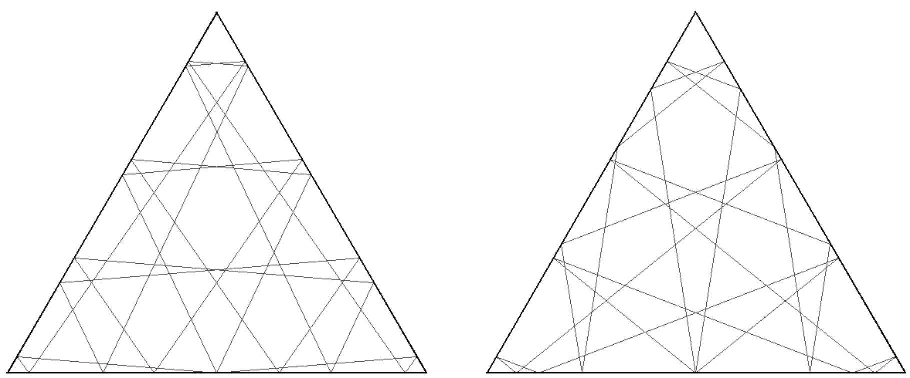

FIGURE 10. Period 22 orbits $(1, 10)$ (left) and $(4, 7)$ (right).

has period $2n$. Note that the period 22 orbits $(1, 10)$ and $(4, 7)$ are not equivalent since they have different lengths and representation angles (see Figures 9 and 10).

To answer to question (2), we reduce the problem to counting partitions by constructing a bijection between classes of orbits with period $2n$ and partitions of n with 2 and 3 as parts. For a positive integer n, a *partition* of n is a nonincreasing sequence of nonnegative integers whose terms sum to n. Such a sequence has finitely many non-zero terms, called the *parts*, followed by infinitely many zeros. Thus, we seek pairs of nonnegative integers (a, b) such that $n = 2a + 3b$. The reader can easily prove:

Lemma 1. *For each $n \in \mathbb{N}$, let*

$$X_n = \left\{(x, y) \in \mathbb{Z}^2 \mid 0 \le x \le y,\ x \equiv y \pmod 3,\ x + y = n\right\}$$

and

$$Y_n = \{(a, b) \in \mathbb{Z}^2 \mid a, b \ge 0 \text{ and } 2a + 3b = n\}.$$

The function $\varphi : Y_n \to X_n$ given by $\varphi(a, b) = (a, a + 3b)$ is a bijection.

Combining Theorem 4 and Lemma 1, we get:

Corollary 4. *For each $n \in \mathbb{N}$, there is a bijection between period $2n$ orbits and the partitions of n with 2 and 3 as parts.*

Counting partitions of n with specified parts is well understood (e.g., Sloane's A103221, [11]). The number of partitions of n with 2 and 3 as parts is the coefficient of x^n in the generating function

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \mathcal{O}(n) x^n \\ &= (1 + x^2 + x^4 + x^6 + \cdots)(1 + x^3 + x^6 + x^9 + \cdots) \\ &= \frac{1}{(1 - x^2)(1 - x^3)}. \end{aligned}$$

To compute this coefficient, let ω be a primitive cube root of unity and perform a partial fractions decomposition. Then

$$\begin{aligned} f(x) &= \frac{1}{4(1+x)} + \frac{1}{4(1-x)} + \frac{1}{6(1-x)^2} + \frac{1}{9}\left(\frac{1+2\omega}{\omega - x} + \frac{1+2\omega^2}{\omega^2 - x}\right) \\ &= \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n x^n + \frac{1}{4}\sum_{n=0}^{\infty} x^n + \frac{1}{6}\sum_{n=0}^{\infty}(n+1)x^n \\ &\quad + \frac{1}{9}\sum_{n=0}^{\infty}(\omega^{2n+2} + 2\omega^{2n} + \omega^{n+1} + 2\omega^n)x^n \end{aligned}$$

and we have

$$\mathcal{O}(n) = \frac{(-1)^n}{4} + \frac{n}{6} + \frac{5}{12} + \frac{1}{9}\left(\omega^{2n+2} + 2\omega^{2n} + \omega^{n+1} + 2\omega^n\right).$$

By easy induction arguments, one can obtain simpler formulations (see [11]):

Theorem 5. *The number of distinct classes of period* $2n$ *is exactly*

$$\begin{aligned}\mathcal{O}(n) &= \begin{cases} \lfloor \frac{n}{6} \rfloor, & n \equiv 1 \pmod 6 \\ \lfloor \frac{n}{6} \rfloor + 1, & \textit{otherwise} \end{cases} \\ &= \left\lfloor \frac{n+2}{2} \right\rfloor - \left\lfloor \frac{n+2}{3} \right\rfloor .\end{aligned}$$

Let us refine this counting formula by counting only primitives. For every divisor d of n, the (n/d)-fold iterate of a primitive period $2d$ orbit has period $2n$. Hence, if $\mathcal{P}(n)$ denotes the number of primitive classes of period $2n$, then

$$\mathcal{O}(n) = \sum_{d|n} \mathcal{P}(d).$$

A formula for $\mathcal{P}(n)$ is a direct consequence of the Möbius inversion formula (see [9]). The Möbius function $\mu : \mathbb{N} \to \{-1, 0, 1\}$ is defined by

$$\mu(d) = \begin{cases} 1, & d = 1 \\ (-1)^r, & d = p_1 p_2 \cdots p_r \text{ for distinct primes } p_i \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 6. *For each* $n \in \mathbb{N}$*, there are exactly*

$$\mathcal{P}(n) = \sum_{d|n} \mu(d) \mathcal{O}(n/d)$$

primitive classes of period $2n$*.*

Theorems 5 and 6 imply:

Corollary 5. $\mathcal{O}(n) = 0$ *if and only if* $n = 1$*;* $\mathcal{P}(n) = 0$ *if and only if* $n = 1, 4, 6, 10$.

Corollary 6. *The following are equivalent:*

(1) *The integer* n *is 1 or prime.*
(2) $\mathcal{P}(n) = \mathcal{O}(n)$*.*
(3) *All classes of period* $2n$ *are primitive.*

Table 1 in the Appendix displays some values of $\mathcal{O}$ and $\mathcal{P}$. The values $\mathcal{O}(4) = 1$ and $\mathcal{P}(4) = 0$, for example, indicate that the single class of period 8 contains 2-fold iterates of the period 4.

We conclude with an example of a primitive class of period $2n$ for each $n \in \mathbb{N} - \{1, 4, 6, 10\}$. But first we need the following self-evident lemma:

Lemma 2. *Given an orbit* $(x, y) \in \Gamma_O$*, let* $d \in \mathbb{N}$ *be the largest value such that* $x/d \equiv y/d \pmod 3$*. Then* (x, y) *is primitive if and only if* $d = 1$*; otherwise* (x, y) *is a* d*-fold iterate of the primitive* $(x/d, y/d)$*.*

Although d is difficult to compute, it is remarkably easy to check for primitivity.

Theorem 7. *An orbit* $(x,y) \in \Gamma_O$ *is primitive if and only if either*

(1) $\gcd(x,y) = 1$ *or*
(2) $(x,y) = (3a,3b)$, $\gcd(a,b) = 1$ *and* $a \not\equiv b \pmod 3$ *for some* $a, b \in \mathbb{N} \cup \{0\}$.

Proof. If $\gcd(x,y) = 1$, the orbit (x,y) is primitive. On the other hand, if $(x,y) = (3a,3b), a \not\equiv b \pmod 3$ and $\gcd(a,b) = 1$ for some a, b, let d be as in Lemma 2. Then $d \neq 3$ since $a \not\equiv b \pmod 3$ But $\gcd(a,b) = 1$ implies $d = 1$ so (x,y) is also primitive when either (1) or (2) holds. □

Example 1. *Using Theorem 7, the reader can check that the following orbits of period* $2n$ *are primitive:*

- $n = 2k+1, k \geq 1 : (k-1, k+2)$
- $n = 2 : (1,1)$
- $n = 4k+4, k \geq 1 : (2k-1, 2k+5)$
- $n = 4k+10, k \geq 1 : (2k-1, 2k+11)$.

Since Corollary 5 asserts that there are *no* primitive orbits of period 2, 8, 12 or 20, Example 1 exhibits a primitive orbit of every possible even period.

5. Concluding Remarks

Many interesting open questions remain; we mention three:

(1) What can be said if the equivalence relation on the set of all periodic orbits defined above were defined more restrictively? For example, one could consider an equivalence relation in which equivalent orbits have cycles of incidence angles that differ by a *cyclic* permutation.

(2) Every isosceles triangle admits a period 4 orbit resembling (1,1) and every acute triangle admits an orbit of period 6 resembling (0,3). Empirical evidence suggests that every acute isosceles triangle with base angle at least 54 degrees admits an orbit of period 10 resembling (1,4). Thus we ask: To what extent do the results above generalize to acute isosceles triangles?

(3) Define $\mathcal{OO}(n) = \sum_{i=1}^{n} \mathcal{O}(n)$ and $\mathcal{PP}(n) = \sum_{i=1}^{n} \mathcal{P}(n)$. Inspection of graphs suggests that $\mathcal{O}(n)$ and $\mathcal{P}(n)$ grow linearly while $\mathcal{OO}(n)$ and $\mathcal{PP}(n)$ are approximately quadratic (see Figures 11 and 12). Dennis DeTurck asked: Are $\mathcal{O}(n)$ and $\mathcal{P}(n)$ in some sense derivatives of $\mathcal{OO}(n)$ and $\mathcal{PP}(n)$? Furthermore, note that $\mathcal{QQ}(n) = \frac{\mathcal{PP}(n)}{\mathcal{OO}(n)} < 1$. Does $\lim_{n\to\infty} \mathcal{QQ}(n)$ exist? If so, this limit is the fraction of all periodic orbits (counted by $\mathcal{O}$) that are primitive.

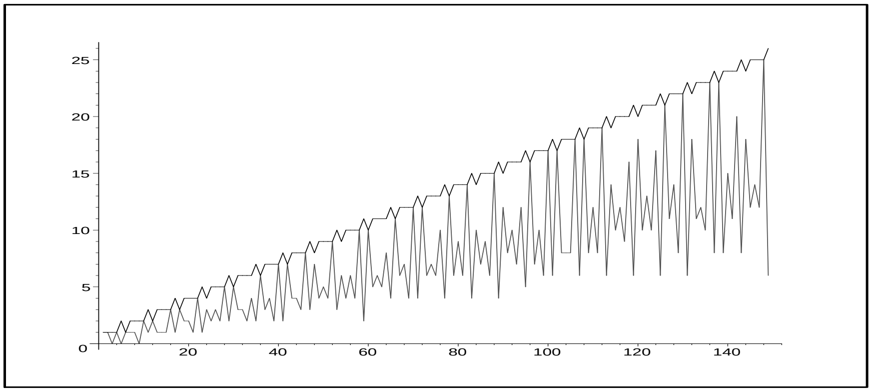

FIGURE 11. Graphs of $\mathcal{O}(n)$ and $\mathcal{P}(n)$

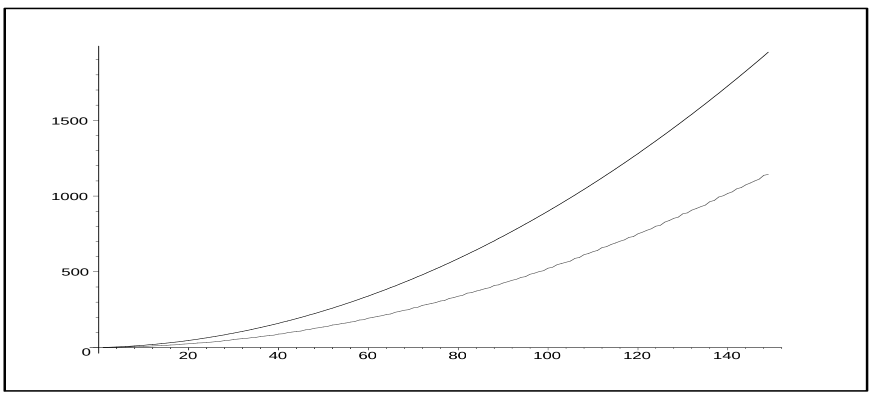

FIGURE 12. Graphs of $\mathcal{OO}(n)$ and $\mathcal{PP}(n)$.

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7. Appendix

n	$2n$	$\mathcal{O}(n)$	$\mathcal{P}(n)$	n	$2n$	$\mathcal{O}(n)$	$\mathcal{P}(n)$
1	2	0	0	31	62	5	5
2	4	1	1	32	64	6	3
3	6	1	1	33	66	6	3
4	8	1	0	34	68	6	2
5	10	1	1	35	70	6	4
6	12	2	0	36	72	7	2
7	14	1	1	37	74	6	6
8	16	2	1	38	76	7	3
9	18	2	1	39	78	7	4
10	20	2	0	40	80	7	2
11	22	2	2	41	82	7	7
12	24	3	1	42	84	8	2
13	26	2	2	43	86	7	7
14	28	3	1	44	88	8	4
15	30	3	1	45	90	8	4
16	32	3	1	46	92	8	3
17	34	3	3	47	94	8	8
18	36	4	1	48	96	9	3
19	38	3	3	49	98	8	7
20	40	4	2	50	100	9	4
21	42	4	2	51	102	9	5
22	44	4	1	52	104	9	4
23	46	4	4	53	106	9	9
24	48	5	1	54	108	10	3
25	50	4	3	55	110	9	6
26	52	5	2	56	112	10	4
27	54	5	3	57	114	10	6
28	56	5	2	58	116	10	4
29	58	5	5	59	118	10	10
30	60	6	2	60	120	11	2

Table 1. Sample Values for $\mathcal{O}(n)$ and $\mathcal{P}(n)$.

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