Math 456: Dynamics and Geometry of Chaos   Fall 2024
Lab 3: Period-doubling and the appearance of chaos
Deadline: 16 December

> The ordinary-sized stuff which is our lives—clouds–daffodils–waterfalls–and what happens in a cup of coffee when the cream goes in—these things are full of mystery, as mysterious to us as the heavens were to the Greeks. We're better at predicting events at the edge of the galaxy or inside the nucleus of an atom than whether it'll rain on auntie's garden party three Sundays from now.
>
> —Tom Stoppard, *Arcadia*

Reminders

- Save your work **every time** you enter new material in a cell and *before* you send it to the kernel for evaluation.
- Your submission should contain responses to the **Notebook exercises**. Every cell (or sequence of cells) in your worksheet should have, at least, a preceding comment that briefly describes the cell's content and what the outcome means.
- You can develop your code and write your report with a partner. Submit one notebook for the group. Include the names of group members at the top of the notebook.
- Submit to the Beachboard dropbox **one** worksheet file named `yourlastname_lab#.nb`. Include the names of all group members.
- To reduce the size of the file, use the Delete all output facility under Cell on the drop-down menu.
- The Wolfram Documentation facility is accessible in the Help menu.
- *Mathematica* code appears in `typewriter font`.

Task

Write a program that graphically indicates the attracting periodic cycles for the logistic family

$$\ell_a(x) = a\,x\,(1-x) \qquad 0 < a \leq 4.$$

Specifics

Following the critical point. Every map ℓ_a has one critical point: $\frac{1}{2}$ ($-\infty$ can be thought of as a superattracting fixed point). The following result tells us that, for $0 < a \leq 4$, ℓ_a can have only one periodic attracting orbit. When $a > 4$, $\frac{1}{2}$ belongs to the "basin of $-\infty$ and there are no attracting periodic points in $[0,1]$.

Fact. *If $(a_1, \ldots, a_m)$ is an attracting periodic cycle for ℓ_a, the critical point $\frac{1}{2}$ belongs to the cycle's basin of attraction.*

Thus, we can detect a *finite* attractor of ℓ_a (if it has one) by examining only the orbit of $c = \frac{1}{2}$ for values of $a \in (0, 4]$.

First, for a given value of a, we compute the first k elements of the trajectory of c:

$$\{c, \ell_a(c), \ell_a^2(c), \ldots, \ell_a^k(c)\}.$$

If we compute a long sequence of iterates, we expect the trajectory to be *near* a periodic attracting cycle (if one exists)

$$(b_1, b_2 = \ell_a(b_1), \ldots, b_n = \ell_a(b_{n-1}), \ell_a(b_n) = b_1).$$

To make a picture of this, we could compute, say, the first 300 elements of the orbit of c, then *ignore* the first 200 elements of the orbit, and finally *plot* the points

$$L_a = (a, \ell_a^{201}(c)), (a, \ell_a^{202}(c)), \ldots, (a, \ell_a^{300}(c)).$$

Here, we're thinking of the horizontal axis as a *parameter space*—where points correspond to a value of the parameter a. The vertical axis corresponds to points in the dynamical space—where the trajectory of c lives. If, for a given value of a, there is a periodic attracting cycle whose length is less than 100, this set of points should indicate it. For instance, when $a = 2$, there is an attracting fixed point (namely, c). Thus, the list L_0 consists of 100 points each of which is very close to $(2, \frac{1}{2})$. As a increases above 3, an attracting cycle

$$(b_1 = \ell_a(b_2), b_2 = \ell_a(b_1))$$

of period two appears. In this case, L_a should be a list of 100 points that are close to the pair

$$(a, b_1), (a, b_2).$$

The procedure For a specific value of a, you can use your iteration procedure to produce the first 300 points in the orbit of c. `Table` or `Take` or `Drop` allow you to select the last 100 iterates. The idea is to start at $a = 0$, compute the list L_0, then increment a by a small amount Δa and compute $L_{0+\Delta a}$, and so on.

To start off, use a "large" value for Δa, say, .1. Once you get the thing working, then you can use smaller values—.001 might be as small as you want to go. For an increment this small, *Mathematica* will likely take several minutes to make the lists.

Now, *Mathematica* doesn't see this list of lists as a bunch of points to be plotted. You can use

`ListPlot[`*`list-of-points`*`]`

to create graphics objects that `Show[`*`lists-of-pointplots`*`]` will display in one coordinate space. (Use a semi-colon at the end of the `ListPlot` line so that *Mathematica* doesn't display these lists.) You can specify the size of a point that *Mathematica* plots by setting the `PlotStyle` option with `PointSize`—in `ListPlot`.

Interpreting the picture You should now have a strange looking picture (called a *bifurcation* diagram) that, starting on the left (at $a = 0$) and moving right, shows a curve that splits into two each branch of which then splits into two, etc. This accounts for the term "period-doubling." Eventually, the image seems to be *full* of points in many places. Remember what this image represents: corresponding to each value of a on the horizontal axis there is a set of "vertical" points that reveal an attracting cycle for the map ℓ_a. The length of this cycle might be very long; so, there might be a large number of points plotted. (Of course, you're limiting the number to 100, but you could use more points.) As an example, for values of a between 0 and 3 there's only one point plotted. This tells you that there's an attracting fixed point for those maps. Also, notice that that "curve of fixed points" indicates that the family ℓ_a is continuous in the parameter a.

Notebook exercises

1) Use the bifurcation diagram to estimate the value of a for which an attracting cycle of length four appears. (Meaning: attracting points that have period four.) You can check this either analytically or by plotting the graph of l_a^2.
2) You should be able to see vertical strips that are somewhat empty. What approximate interval (α, β) of a values corresponds to the most prominent such "window?" Using the same step-size Δa as before (.001 should be adequate), plot a bifurcation diagram *on* (α, β). That is, start with $a = \alpha$ and repeatedly increment a by Δa until you reach β. Interpret the resulting picture. What does it tell you about the maps for values of a between α and β?
3) By zooming in on the original bifurcation diagram, find a value of a for which ℓ_a has an attractor with period three. You can find an approximate value for a when a 3-cycle first appears by animating the graph of ℓ_a^3 and looking for the appearance of a fixed point. By magnifying the bifurcation diagram at this value of a, estimate the range of a values for the period-doubling cascade that follows on from this 3-cycle—that is, the range for cascade of attracting cycles of length $2^k 3$.
4) Zoom in to estimate the value of a where the attracting 2^k-periodic points accumulate. What must happen in the family ℓ_a before the appearance of the next cascade of attracting periodic cycles? What period of attracting cycle appears *just after* powers-of-2 cycles?