Math 456: Dynamics and Geometry of Chaos — Fall 2024
Lab 2: Cobweb plots
Deadline: 8 November

> "I shouldn't be surprised if it hailed a good deal tomorrow," Eeyore was saying. "Blizzards and whatnot. Being fine today doesn't Mean Anything. It has no sig— what's that word? Well, it has none of that. It's just a small piece of weather."
> —A. A. Milne, *The House at Pooh Corner*

Reminders

- Save your work **every time** you enter new material in a cell and *before* you send it to the kernel for evaluation.
- Your submission should contain responses to the **Notebook exercises**. Every cell (or sequence of cells) in your worksheet should have, at least, a preceding comment that briefly describes the cell's content and what the outcome means.
- You can develop your code and write your report with a partner. Submit one notebook for the group. Include the names of group members at the top of the notebook.
- Submit by email **one** notebook file named `yourlastname_lab#.nb`
- To reduce the size of the file, use the Delete all output facility under Cell on the drop-down menu.
- The Wolfram Documentation facility is accessible in the Help menu.
- *Mathematica* code appears in `typewriter font`.

Task. Write a program that will plot a cobweb diagram for a one-dimensional map and an arbitrary starting point.

Plotting a collection of points in the plane. You can represent a list of points

$$\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$$

as

```
L = { {x[1],y[1]}, {x[2],y[2]}, ..., {x[n],y[n]} } .
```

Note the use of the square brackets to indicate the subscript. Now you can create a *graphics object* using

```
Lplot = ListPlot[L]
```

and display the plot with

```
Show[Lplot] .
```

Try plotting the vertices of a square: points at $(1,1), (-1,1), (-1,-1), (1,-1)$.

You can use various styles of axes, adjust the size and color of the points that *Mathematica* draws as well as some other features. One useful one is to set the horizontal and vertical scales to be the same. To do this, set the option `AspectRatio->Automatic`. See ListPlot for options.

To connect the points in a list, set the option

`Joined->True`.

Try connecting the vertices of the square above.

Plotting the graph of a map. For a map $f(x)$, the command is simple. The command

```
Plot[f[x], {x,xmin,xmax}, options]
```

creates and displays the graph on the interval $(xmin, xmax)$. For example,

```
Plot[x, {x,-1,2}]
```

plots the diagonal line $y = x$ on $[-1, 2]$.

To get the hang of it, plot the graphs of the following maps on the specified intervals

1) $x \longrightarrow -\frac{1}{4}x + \frac{1}{4} \qquad [-3, 4]$
2) $x \longrightarrow x^2 - \frac{1}{4} \qquad [-2, 2]$
3) $x \longrightarrow x(1 - x) \qquad [0, 1]$

You can name a plot—e.g.,

```
idPlot = Plot[x, {x,-1,2}, options]     fPlot = Plot[f[x], {x,-1,2} options].
```

By setting a color option—`PlotStyle->Blue`, say—the graph will be rendered in the specified color. Now you can superimpose the two with

```
Show[{idPlot,fPlot}, options].
```

Pick a function for f and display the graph-plots above on one set of axes while giving each graph a different color.

From an orbit to a cobweb plot. A cobweb diagram consists of an ordered list of points that are joined by either vertical or horizontal line segments. Say you have a map $f(x)$ and a starting point x_0. Work out (on paper) the list of points that get connected by segments. Which pairs have a vertical segment between them? Which have a horizontal connector? Express these in terms of x_0 and its iterates under f. For instance, the cobweb begins with the point (x_0, x_0) on the diagonal line $y = x$. It then extends vertically to the point $(x_0, f(x_0))$. What's the next move? The one after that, ...?

You can use your iteration routine to compute the trajectory of x_0 and then place the trajectory values into the appropriate places in a list of cobweb points. For example, say you have a 100-point orbit

$$x_0,\ x_1 = f(x_0),\ \ldots\ , x_{99}.$$

The first cobweb point would be (x_0, x_0) while the second would be (x_0, x_1) and so on.

Once you have the appropriate list of cobweb points, it's simply a matter of connect-the-dots. As in the iteration case, try to write a plotting routine so that it takes as input 1) a map $f(x)$,

2) a starting point x_0, and 3) a number of iterations. You might find it convenient to "merge" two lists. The `Table` and `Flatten` commands can produce this result.

To complete the cobweb picture, superimpose the plots of 1) the map $f(x)$, 2) the diagonal line $y = x$, and 3) the cobweb segments. Remember to name the plot of the cobweb segments. *Note*: Depending upon your choice of x_0, you might have to adjust the endpoints of the graph-plots. As an option to `Show` you can use `PlotRange -> {{xmin,xmax},{ymin,ymax}}`. The location of the axes can be specified by `PlotOrigin -> {a,b}` so that a horizontal axis appears at $y = b$ and a vertical axis at $x = a$.

Notebook exercises

Make cobweb diagrams for the following maps in the logistic family

$$x \longrightarrow a\,x(1-x).$$

Using a variety of starting points, look for attracting periodic points—what do you look for (visually)? Are there trajectories that *aren't* attracted somewhere? If so, how does the corresponding diagram appear? Decide (conjecture) what the global dynamics is. Where are the attracting points? Which points do the respective attracting points attract—what's the *basin of attraction*?

1) $x \longrightarrow x\,(1-x)$ on $0 \le x \le 1$

2) $x \longrightarrow 2\,x\,(1-x)$ on $0 \le x \le 1$

3) $x \longrightarrow 4\,x\,(1-x)$ on $0 \le x \le 1$
   Use cobweb plots in different colors to illustrate sensitive dependence on initial conditions.

Compare cases 4-6 below. How does the dynamical behavior change with the coefficient?

4) $x \longrightarrow 2.9\,x\,(1-x)$ on $0 \le x \le 1$

5) $x \longrightarrow 3\,x\,(1-x)$ on $0 \le x \le 1$

6) $x \longrightarrow 3.1\,x\,(1-x)$ on $0 \le x \le 1$

For amusement (not to submit). Consider the map

$$x \longrightarrow \frac{1+\cos\left(\pi\,(2\,x-1)\right)}{2}$$

which maps $[0,1]$ onto $[0,1]$. Explore trajectories of various points in $[0,1]$. Compare the dynamics of this map to that of the logistic map

$$x \longrightarrow 4\,x(1-x).$$