Exam 3 Math 444 Fall 2024 Due: Wednesday, 18 Dec

- Write concisely and clearly. Submit your work to the Canvas drop box.
- You can work with a partner–submit one paper for the group.
- Papers written in *LaTeX* will receive a 5% bonus.
- *Ethic*: In completing this exam, you should not consult other sources—excepting the text and instructor. The work represented by what you (and your group) submit should be entirely your own.

1 Orthogonal matrices (10 points)

A square matrix A is *orthogonal* when $AA^T = I$ where A^T is the transpose of A and I is the identity matrix. Let O_n be the set of $n \times n$ orthogonal matrices.

a) Establish that O_n is a subgroup of the invertible $n \times n$ matrices GL_n.

b) Show that the determinant of an orthogonal matrix is either 1 or -1.

   Suggestion: Recall the homomorphism property of the determinant:

$$\det(AB) = \det A \det B.$$

c) Let SO_n be the orthogonal matrices whose determinant is 1. Show that SO_n is a *normal* subgroup of O_n.

d) Describe the quotient group $\mathrm{O}_n/\mathrm{SO}_n$. Use the first isomorphism theorem to find a group to which the quotient group is isomorphic.

2 Groups of order p^2 when p is prime (20 points)

Let p be a prime number and let G be a group with $|G| = p^2$. The task is to show that G is either cyclic or isomorphic to $\mathbf{Z}_p \times \mathbf{Z}_p$. So, suppose that G is not cyclic.

a) Let $a \neq e$ be some element of G and $A = \langle a \rangle$. What's the order of A?

b) Consider the cosets of A:

$$G/A = \{A, g_2A, \ldots, g_nA\}.$$

   What's the value of n?

c) Let $\phi : G \to S_n$ (S_n is the symmetric group) be defined by

$$\phi(x)(g_kA) = xg_kA.$$

   Show that ϕ is a homomorphism and that $\phi(x)$ is a permutation on G/A for all $x \in G$.

d) Argue that the only two possibilities for the kernel of ϕ are 1) $\ker\phi = A$ or 2) $\ker\phi = \{e\}$.

   But, case 2) implies that ϕ is injective. Compare the order of $\phi(G)$ to the order of S_n and show that ϕ can't be injective.

   Now conclude that A is a normal subgroup of G.

e) Let $b \in G - A$. What's the order of b? Since A is normal,

$$bab^{-1} = a^k \tag{1}$$

for some k such that $1 \leq k \leq |a| - 1$.

f) By a similar line of reasoning, the subgroup $B = \langle b \rangle$ is also normal in G so that for some m such that $1 \leq m \leq |b| - 1$,

$$aba^{-1} = b^m. \tag{2}$$

Recall the orders of $\langle a \rangle$ and $\langle b \rangle$. Use equations (1) and (2) to derive

$$a^{k-1}b^{m-1} = e.$$

Why does this condition imply that

$$k - 1 = m - 1 = 0?$$

Now, conclude that $ab = ba$.

g) By considering *all* of the products $a^i b^j$ exhibit an isomorphism so that $G \simeq \mathbf{Z}_p \times \mathbf{Z}_p$.