Exam 2 Math 444 Fall 2024
Due: Friday, 8 November

- Write concisely and clearly. Submit your work to the Canvas drop box.
- You can work with a partner–submit one paper for the group.
- Papers written in *LaTeX* will receive a 5% bonus.
- *Ethic*: In completing this exam, you should not consult other sources—excepting the text and instructor. The work represented by what you (and your group) submit should be entirely your own.

1 A group that's a collection of sets

Let S be a set of things and let P be the set of subsets of S. For $A, B \in P$, define

$$A * B = ((S - A) \cap B) \cup (A \cap (S - B)).$$

a) Show that
$$A * B = (A \cup B) - (A \cap B).$$
A Venn diagram suffices.

b) Show that $(P, *)$ is a commutative group. What's the group identity? Given $A \subset S$, what's the inverse of A?

c) Consider the set
$$S = \{\text{Alice, Bob, Carol, Don, Erin, Frank, Gary, Harriot}\}$$
Using the set operation $*$ find the subgroup $(Q, *)$ of $(P, *)$ generated by the sets
$$\{\text{Alice, Bob}\},\ \{\text{Carol, Don}\},\ \{\text{Erin, Frank}\},\ \{\text{Gary, Harriot}\}.$$

d) Express Q as a direct product of cyclic groups.

2 Automorphisms and a group's center

Let $Z(G)$ be the center of G. Show that $x \in Z(G)$ if and only if the inner automorphism ϕ given by $\phi(x) = gxg^{-1}$ is the identity automorphism.

3 Groups of order six

Describe *all* groups of order six. (Two groups are equivalent if they're isomorphic.) Make sure that your approach is sufficiently systematic to establish your list as *complete*. Make your arguments concise.

Suggestion: Explore what it takes to construct a group multiplication table.

4 Normal subgroups of normal subgroups

Consider the matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

a) Using a multiplication table, show that the eight matrices

$$\pm I, \ \pm A, \ \pm B, \ \pm C$$

form a group G.

Note: Since you know immediately how products that involve $\pm I$ behave, your table can treat just the three elements A, B, and C. Also, matrix multiplication amounts to function composition and so, is associative.

b) Show that $\langle -I, A \rangle$ is normal in G.

c) Show that $\langle -I \rangle$ is normal in $\langle -I, A \rangle$, but is *not* normal in G.