Exam 2 Math 362 Spring 2025 Due: 11 May

- Write concisely and clearly. Submit your work to the Canvas drop box.
- You can work with a partner–submit one paper for the group.
- Papers written in *LaTeX* will receive a 5% bonus.
- *Ethic*: In completing this paper you should not consult other sources—excepting the text and instructor. The work represented by what you write and submit should be entirely your own.

1 The map $z \longrightarrow \frac{1}{2}(z + \frac{1}{z})$

Consider the map

$$f(z) = \frac{1}{2}(z + \tfrac{1}{z}).$$

a) Show that$f$ has three fixed points on the complex sphere $\widehat{\mathbf{C}}$?

b) With a mobius transformation $w = Tz$, make a change of coordinates that moves the fixed points in $\mathbf{C}$ to 0 and ∞. Express the map in the w coordinates and make a diagram of the coordinate change.

c) Use the geometric properties of mobius transformations and your understanding of the map in the w space to describe the map in the z space. Pay particular attention to the map's behavior relative to the fixed points. Also, look for special families of sets that f preserves *as a family*. What does f do to the members of these families?

2 When the complex derivative is zero

a) Let $U \subset \mathbf{C}$ be an open set and suppose that for some $a \in \mathbf{C}$, $f(z) = a$ for all $z \in U$. Use the amplitwist description of the derivative to show that $f'(z) = 0$ on U. Now show this analytically—that is, from the limit definition of derivative.

b) A subset U of $\mathbf{C}$ is *connected* when for every pair of points a, b in U, there is a path from a to b that lies entirely in U.

   Suppose that f is analytic and $f'(z) = 0$ on a connected, open set $U \subset \mathbf{C}$. Use the geometry of amplitwisting to show that $f(z)$ is constant on U. Now show this analytically—from the limit definition of derivative.

c) Show that the condition that U is connected is necessary.

3 Mapping lines to lines and circles to circles

Recall that an *entire* function is analytic on $\mathbf{C}$.

a) Describe *all* entire functions that send vertical lines to vertical lines.

   Suggestion: Consider what this condition tells you about the derivative (as an amplitwist) of such a function.

b) Express a function that conformally sends the family of vertical lines to the family F of concentric circles about 0.

c) Describe *all* entire functions that preserve the members of F (that is, sends a circle centered at 0 to a circle centered at 0).