Midterm study guide — Math 224 — Spring 2024

Exam items will be selected from assigned exercises and the following.

1) Consider a quadrilateral (4-sided polygon) with corners at A, B, C, D. Connect the midpoints of *adjacent* sides to form a new quadrilateral figure.

   a) Draw a diagram of the situation.
   b) Use vector considerations to show that the new quadrilateral is a parallelogram.

2) Here are four planes A, B, C, D:

$$\begin{array}{ll} A:\ x+2y+2z=4 & B:\ x-z=1 \\ C:\ x+y=1 & D:\ y-z=0. \end{array}$$

   a) Sketch the planes A and B and their intersection. Sketch the planes C and D and their intersection.
   b) Give parametric equations for the line L formed by the intersection of A and B. Give parametric equations for the line M formed by the intersection of C and D.
   c) What do we mean when we say that there's a *distance between two lines*?
   d) Find the distance between L and M.

3) Suppose a particle P is travelling *at constant speed* v along the line defined by $y = 0$ and $z = 1$. It's moving in the positive x direction. Suppose a light source is located at $(0, 0, 2)$.

   a) Parametrize the particle's path with a time parameter t so that, when $t = 0$, P is at $(0, 0, 1)$.
   b) Use t to parametrize the shadow S of P that falls on the xy-plane (where $z = 0$). Express the velocity and speed of S at any time t.
      Suggestion: A sketch would be a good idea. Look for similar triangles.
   c) Use t to parametrize the shadow T of P that falls on the plane given by $x + 2z = 0$. Express the velocity and speed of T at any time t.
   d) Does the shadow remain on the plane for all t?
   e) If not, when does it "leave" the plane and what is its speed at the moment it leaves?

4) With numbers, if $ab = ac$ and $a \neq 0$, then $b = c$. Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be space vectors.

   a) Suppose $\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{w}$ and $\mathbf{u} \neq \mathbf{0}$. Does $\mathbf{v} = \mathbf{w}$ *always*? Provide a reason for your answer.
   b) Suppose $\mathbf{u}\times\mathbf{v} = \mathbf{u}\times\mathbf{w}$ and $\mathbf{u} \neq \mathbf{0}$. Does $\mathbf{v} = \mathbf{w}$ *always*? Provide a reason for your answer.
   c) What can you conclude if both $\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{w}$ and $\mathbf{u}\times\mathbf{v} = \mathbf{u}\times\mathbf{w}$.

5) Consider the function

$$f(x, y) = (\sin x)(\sin y).$$

   a) Sketch the graph of $z = f(x, y)$ on the square $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.
   b) Determine the equation of the plane tangent to the graph when $(x, y) = (0, 0)$.
   c) Describe the intersection of the tangent plane at $(0, 0, 0)$ with the graph of $z = f(x, y)$.
   d) At what points on the square does the graph have a tangent plane that's horizontal?

6) Consider the curve C in the plane given by

$$x^y = y^x.$$

a) Find the slope $\frac{dy}{dx}$ of the line tangent to C at a point (a, b) on the curve. Express the slope in terms of a and b.
b) Note that if (a, b) is on C, so is (b, a). Express the slope of the line tangent to C at (b, a).
c) Determine the relationship between the slopes at (a, b) and at (b, a).

7) Suppose a particle of mass m falls due to the vertical downward force of gravity **along a chord** of a circle as depicted in Figure 1. (Think of a bead sliding along a tight wire.) Assume that the circle has radius one, is centered at $(0, 0)$, and take B to be $(0, -1)$.

a) Use polar coordinates to describe A and express the vector $\mathbf{u}$ from A to B.
b) Project the acceleration vector due to gravity (use g as the magnitude of acceleration) onto $\mathbf{u}$.
c) Work out the velocity $\mathbf{v}$ and position $\mathbf{r}$ vector functions for the particle.
d) Determine the time T it takes for the particle to go from A to B.
e) Does T depend on A's location? What do you conclude from this result?

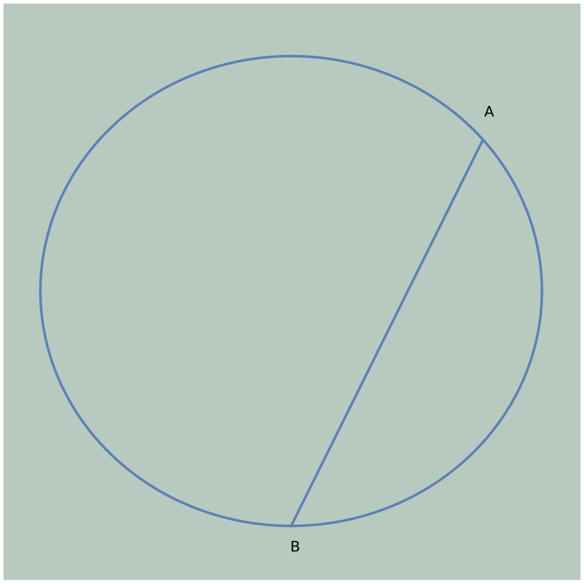


Figure 1: A particle falls from A to B

8) The curvature κ of a curve $\mathbf{r}(t)$ at t is given by

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right|$$

where $\mathbf{T}$ is the unit tangent vector and s is the arclength parameter. Taking $\mathbf{N}$ to be the unit normal to the curve, derive the following equation

$$\frac{d\mathbf{T}}{ds} = \kappa \mathbf{N}.$$

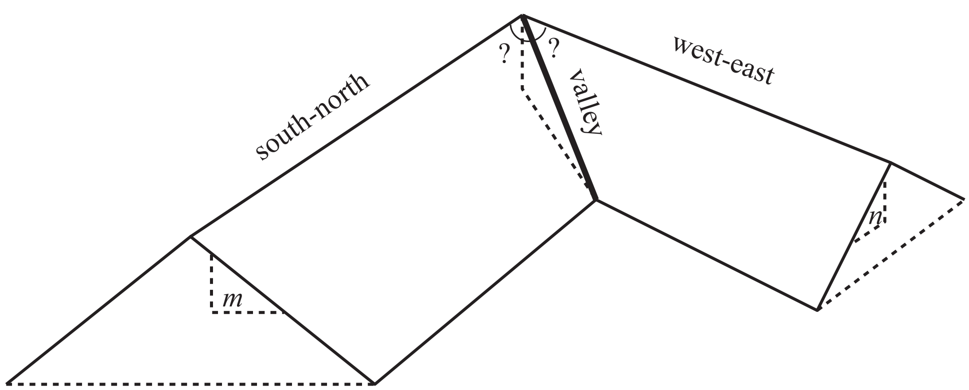
9) Two planes A and B are flying at constant velocity at the same constant altitude. At times $t_1 = 0$ and $t_2 = 1$ their respective positions are

$$A : (3,1) \text{ and } (2,5) \qquad (1,2) \text{ and } (5,3).$$

a) Parametrize each plane's path with respect to time.
b) At what time are the two planes closest to one another? (Not the paths, but the planes themselves.)
c) How far apart are the planes at their closest?

10) A house has two sections of roof that run perpendicular to one another (this means that the lines of the ridges are perpendicular). The line segment where the two roofs intersect is called the *valley*. The slope (measured in a vertical plane) of the north-south roof is m and the slope of the east-west roof is n.

a) Think of a roof as a plane and choose a convenient system of coordinates in order to find the vectors that are normal to each roof-plane.
b) Find a vector *in the direction of the valley.*
c) Find the "slope of the valley" as a function of m and n. (Measure slope in the vertical plane that contains the valley.)
d) What angles do the valley and the the two ridge lines form? That is, what angles are formed *on the roof*? Express the angles as a function of m and n.
e) Suppose the slope of each roof is $\frac{5}{12}$. A carpenter's rule-of-thumb is that the slope of the valley is $\frac{5}{17}$. Is this a good rule?



Answers:

2) b) one way: $L : s\left\langle 1, -\frac{3}{2}, 1\right\rangle + \langle 0, 3, -1\rangle \qquad M : t\langle -1, 1, 1\rangle + \langle 1, 0, 0\rangle \qquad$ d) $\dfrac{6}{\sqrt{42}}$

3) a) $\langle vt, 0, 1\rangle \qquad$ b) $\langle 2vt, 0, 0\rangle \qquad$ c) $\left\langle \dfrac{4vt}{vt-2}, 0, \dfrac{2vt}{vt-2}\right\rangle \qquad$ d) no $\qquad$ e) ∞

6) a) $\left.\dfrac{dy}{dx}\right|_{(x,y)=(a,b)} = \dfrac{b^2\,(\ln a - 1)}{a^2\,(\ln b - 1)}$ (or any equivalent expression)

b) $\left.\dfrac{dy}{dx}\right|_{(x,y)=(b,a)} = \dfrac{a^2\,(\ln b - 1)}{b^2\,(\ln a - 1)}$

9) a) $\mathbf{r}_A = \langle 3-t, 1+4t\rangle \quad \mathbf{r}_B = \langle 1+4t, 2+t\rangle \qquad$ b) $t = \frac{3}{10} \qquad$ c) $\frac{1}{\sqrt{2}}$